Chapter 1

Curves

Curves, in the context of geometry, are one-dimensional lines which may be bent. Such a geometric object can describe a thin wire in space or the path of a moving object. Abstractly speaking a curve is a set of points that can be continuously parameterized by a single variable. In the moving object path example, the parameter would be interpreted as time.

A curve is mathematically defined through a continuous function $\gamma\colon I \to S$ (Figure 1.1), where the domain $I = [a, b] \subset \mathbb{R}$ is some interval on real line called the **parameter space**; the target space S is some space (a topological space, the weakest type of point set on which one can define continuity). A **closed curve** is a curve $\gamma\colon \mathbb{T}^1 \to S$ defined on a periodic domain $\mathbb{T}^1 = (\mathbb{R} \bmod 2\pi)$. A **discrete curve** is a curve $\gamma\colon \mathsf{I} \to S$ defined on an (ordered) sequence of indices $\mathsf{I} = (0, 1, 2, \ldots, n-1)$, forming a polygonal image in the target space S. In this chapter, we focus mainly on 2D planar curves, $S = \mathbb{R}^2$, and 3D space curves, $S = \mathbb{R}^3$. Spaces like $\mathbb{R}^2$ and $\mathbb{R}^3$ are particularly equipped with familiar notions of measurements of distances and rotation angles *etc.*, and that induces important geometric measurements for the curves.

Geometric quantities As we study planar and space curves, we follow the philosophy that a **geometric quantity** is a quantity one can measure independent of the parameterization. That is, a geometric quantity is unchanged under re-parameterization. In the planar curve section we introduce tangent vector, normal vector and curvature, which are local geometric quantities. Total length and total turning number of the curve are global geometric quantities.

For space curves we introduce **frames**, which are varying 3D coordinate axes that align with the orientation of the curve. A typical picture of a framed curve is that the frames are the guidelines for anisotropic thickening of the space curve (Figure 1.2). In this sense, a frame field is an additional geometry sitting on top of the infinitesimally thin curve. There are geometric quantities for the framed curves that depend on the choice of frame, such as how the frame

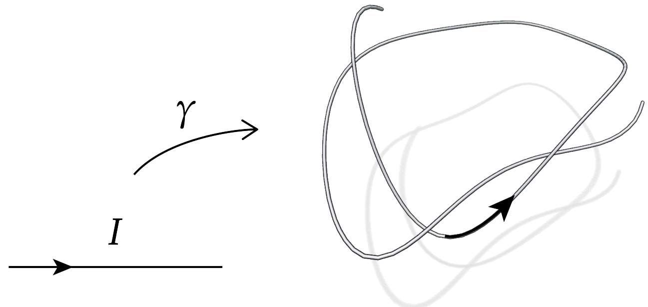


Figure 1.1: *A curve is a continuous map γ from an interval to an ambient space.*

is twisted. There are also geometric quantities, such as the curvature and torsion of the space curve, that are independent of the choice of frames.

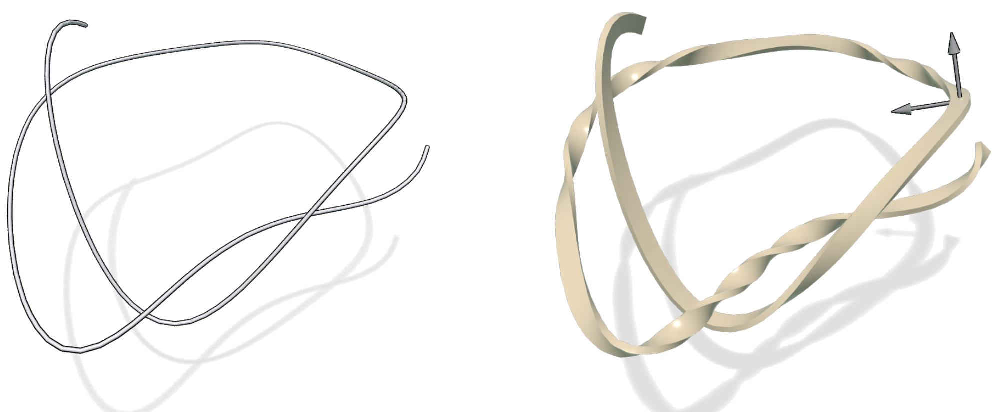


Figure 1.2: *A space curve (left) and a framed space curve (right)*.

Discrete analogy For discrete planar and space curves, we aim to define geometric quantities that are analogous to those of smooth curves. With the smooth theory as the blueprint, one often finds analogous properties and theorems which hold true exactly for the discrete geometric quantities. Distinguish this from a direct discretization of the quantities of interest through some numerical analysis procedure based on approximation theory – in most cases, even for high-order methods, analogous properties and theorems do not hold exactly. Seeking a discrete analog is the core spirit of **discrete differential geometry**. We will see examples where there are several different discrete geometric quantities that correspond to the same smooth counterpart depending on the context in which they are used/defined. In this sense research in discrete differential geometry actually reveals deeper structures than smooth differential geometry alone would observe.

1.1 Planar Curves

A planar curve is a continuous function $\gamma\colon I \to \mathbb{R}^2$ for some interval $I = [a, b] \subset \mathbb{R}$ as the parameter space. Interpreting the parameter $t \in I$ as time, we may imagine $\gamma(t)$ as the position of a particle moving in the plane. The particle path, *i.e.* the image $\gamma(I)$, would then be our geometry of interest.

A re-parameterization of the curve corresponds to traversing the same path at different, even varying, speed. To describe re-parameterization mathematically, let $I = [a, b]$ and $\tilde{I} = [\tilde{a}, \tilde{b}]$ be two intervals and $\gamma\colon I \to \mathbb{R}^2$ our curve. Then $\tilde{\gamma}\colon \tilde{I} \to \mathbb{R}^2$ is a re-parameterization if and only if there is a strictly increasing function $\varphi\colon I \to \tilde{I}$, $\varphi' > 0$, (and so there is an inverse $\varphi^{-1}\colon \tilde{I} \to I$) such that $\gamma = \tilde{\gamma} \circ \varphi$ and $\tilde{\gamma} = \gamma \circ \varphi^{-1}$.

One can look at the velocity of the particle under re-parameterization. The velocity of the first particle is $\frac{d}{dt}\gamma =: \gamma'\colon I \to \mathbb{R}^2$ with speed $|\gamma'|\colon I \to \mathbb{R}$. The velocity of the other particle is $\tilde{\gamma}' = (\gamma \circ \varphi^{-1})' = \gamma' \cdot \frac{1}{\varphi'}$ with speed $|\tilde{\gamma}'| = \frac{1}{\varphi'}|\gamma'|$. This says that both velocity and speed depend on the parameterization, which suggests that neither the velocity nor the speed is a geometric quantity of the particle path. However, normalized velocity $\gamma'/|\gamma'|$ is unchanged under re-parameterization since $\tilde{\gamma}'/|\tilde{\gamma}'| = \gamma'/|\gamma'|$. The normalized velocity is a geometric quantity of the path called the **tangent vector**. To prevent any trouble with a vanishing denominator, we always consider only the case $|\gamma'| \neq 0$.

Definition 1.1 (Regular curve)**.** A **regular planar curve** is a planar curve $\gamma\colon I \to \mathbb{R}^2$ with non-vanishing $|\gamma'|$.

We denote a normalized vector in $\mathbb{R}^2$, *i.e.* a direction, as an element of the unit circle $\mathbb{S}^1 = \{\mathbf{u} \in \mathbb{R}^2 \,|\, |\mathbf{u}| = 1\}$.

Definition 1.2 (Tangent vector)**.** The **tangent vector** $T : I \to \mathbb{S}^1$ of a regular planar curve $\gamma\colon I \to \mathbb{R}^2$ is defined by

$$T(t) := \frac{\gamma'(t)}{|\gamma'(t)|}, \quad t \in I.$$

A normal vector is a direction that is perpendicular to T. There are two conventions of "the" normal vector of a planar curve: 90° counterclockwise rotation of T or 90° clockwise rotation of T. In order to have a closed curve traveled counterclockwise (positively oriented) to have its normal being the "outer normal", we choose the latter convention.

Definition 1.3 (Normal vector)**.** The **normal vector** $N\colon I \to \mathbb{S}^1$ of a regular planar curve is the 90° clockwise rotation of T. That is, if $T = (T_x, T_y)$, then $N = (T_y, -T_x)$.

Note that since the normal vector is purely expressed in terms of the tangent vector, it is a geometric quantity independent of the parameterization.

1.1.1 Arclength Parameterization

The arclength of a curve $\gamma\colon [a,b] \to \mathbb{R}^2$ is calculated by integrating the speed over a period of time

$$L = \int_a^b |\gamma'(t)|\, dt.$$

A parameterized curve γ is said to be arclength-parameterized if γ is moving at unit speed.

Definition 1.4. A curve γ is **parameterized by arclength** if $|\gamma'| = 1$.

Given an arclength-parameterized curve $\gamma\colon [0,L] \to \mathbb{R}^2$, the arclength of the interval $[t_1, t_2]$ is $\int_{t_1}^{t_2} |\gamma'(t)|\, dt = \int_{t_1}^{t_2} 1\, dt = t_2 - t_1$, which is just the length of the interval in parameter space. Thus an arclength parameterization is also called an **isometric parameterization**.

Although an arclength parameterization seems to be just a special case parameterization, it plays a central role in finding further geometric quantities. First of all, any parameterized regular curve $\tilde{\gamma}\colon [a,b] \to \mathbb{R}^2$ can be re-parameterized into an arclength-parameterized curve:

Lemma 1.1. *Let $\tilde{\gamma}\colon [a,b] \to \mathbb{R}^2$ be a regular curve. Then it can be re-parameterized by arclength.*

Proof. Let $L = \int_a^b |\tilde{\gamma}'(t)|\, dt$ be the total arclength. Consider $\varphi\colon [0,L] \to [a,b]$ given by

$$\varphi(t) := a + \int_0^t \frac{1}{|\tilde{\gamma}'(\tilde{t})|}\, d\tilde{t}.$$

Then $\gamma\colon [0,L] \to \mathbb{R}^2$ defined by $\gamma = \tilde{\gamma} \circ \varphi$ is arclength-parameterized. One checks that $|\gamma'| = |\tilde{\gamma}'||\varphi'| = |\tilde{\gamma}'|\frac{1}{|\tilde{\gamma}'|} = 1$. □

The greatest advantage of an arclength parameterization is simply that it is a canonical parameterization. If we define a quantity by first re-parameterizing by arclength and then define the quantity through arclength parameterization, the quantity would be independent of the original parameterization automatically.

For example, it is enough to define the tangent vector T by $T = \gamma'$ for an arclength-parameterized γ.

From now on, unless explicitly mentioned, we consider all curves as arclength parameterized.

1.1.2 Curvature

Curvature measures how much the curve deviates from a straight line. If the tangent T and the normal N stay constant along the curve, we know the curve does not change direction and is thus a straight line. This motivates us to define curvature as the rate of change of T resp. N. Again we consider γ to be arclength-parameterized.

Note that N' must be orthogonal to N since taking the derivative of $\langle N, N\rangle = 1$ gives $2\langle N', N\rangle = 0$. Hence N' is a multiple of T.

Definition 1.5 (Curvature)**.** For an arclength-parameterized curve $\gamma\colon [0, L] \to \mathbb{R}^2$, the **curvature** $\kappa\colon [0, L] \to \mathbb{R}$ is defined as

$$\kappa := \langle N', T\rangle,$$

that is

$$N' = \kappa T.$$

Exercise 1.1. Show that $\gamma'' = T' = -\kappa N$.

Example 1.1 (Curvature of a circle)**.** Consider a circle centered at origin with radius R. Its arclength parameterization is given by

$$\gamma(t) = \begin{bmatrix} R\cos\left(\frac{t}{R}\right) \\ R\sin\left(\frac{t}{R}\right) \end{bmatrix}.$$

The tangent and normal vectors are

$$\gamma'(t) = T(t) = \begin{bmatrix} -\sin\left(\frac{t}{R}\right) \\ \cos\left(\frac{t}{R}\right) \end{bmatrix}, \quad N(t) = \begin{bmatrix} \cos\left(\frac{t}{R}\right) \\ \sin\left(\frac{t}{R}\right) \end{bmatrix},$$

and one sees

$$N'(t) = \frac{1}{R}T(t).$$

Hence the circle has a constant curvature $\kappa = \frac{1}{R}$.

Due to the example above, the reciprocal of curvature $\frac{1}{\kappa}$ is known as the **radius of curvature**. The **center of curvature** is $\gamma(t) - \frac{1}{\kappa(t)}N(t)$ is the center of the circle approximating the curve about $\gamma(t)$. This best-approximating circle of the curve is known as the **osculating circle** (from Latin "circulus osculans:" kissing circle).

1.1.3 Variational approach to model Curvature

Above we have seen curvature defined as the rate of change of the normal vector per unit of arclength. We have also seen that curvature is the reciprocal radius of the osculating circle. In this section we will see yet another approach to curvature based on a **variational principle**.

So, what is a straight line? If we characterize straight lines as curves with constant tangent and normal, then it is natural to define curvature as the rate of change of the normal (or tangent). If we characterize straight lines by saying that the best fitting circle has infinite radius, we model curvature as the reciprocal of the radius of curvature. What if we characterize straight lines as the "shortest curve" connecting two end points? Then we model the curvature as the physical force which tries to straighten a bent curve. Using length as the potential energy, curvature becomes the **tension force**, which is the negative gradient of the potential energy. Let's make this concrete.

Let $p, q \in \mathbb{R}^2$ be two fixed points, and suppose each $\varepsilon \geq 0$ smoothly corresponds to a curve $\gamma_\varepsilon \colon [0, L_0] \to \mathbb{R}^2$ which has $\gamma_\varepsilon(0) = p$ and $\gamma_\varepsilon(L_0) = q$. This is a one-parameter (ε) family of curves. Suppose γ_0 is arclength parameterized, but not necessarily for other $\varepsilon > 0$. The motion of the curve induced by varying ε is called the **variation**, defined by $V(t) = \frac{\partial}{\partial \varepsilon}\gamma_\varepsilon(t)\big|_{\varepsilon=0} \in \mathbb{R}^2$. Our goal is to find the variation γ_ε which decrease the total length of the curve at the fastest rate.

Note that we have $V(0) = V(L_0) = 0$ because we fix the end points.

Now, the total length $L_\varepsilon := \int_0^{L_0} |\gamma_\varepsilon'(t)|\, dt = \int_0^{L_0} \langle \gamma_\varepsilon'(t), \gamma_\varepsilon'(t)\rangle^{1/2}\, dt$ has its variation

$$\begin{aligned}
\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} L_\varepsilon &= \int_0^{L_0} \left\langle \frac{\gamma_\varepsilon'(t)}{|\gamma_\varepsilon'(t)|}, \frac{\partial \gamma_\varepsilon'(t)}{\partial \varepsilon} \right\rangle \bigg|_{\varepsilon=0} dt \\
&= \int_0^{L_0} \langle \gamma'(t), V'(t)\rangle\, dt
\end{aligned}$$

$$= -\int_0^{L_0} \langle \gamma''(t), V(t) \rangle \, dt$$
$$= \int_0^{L_0} \langle \kappa(t) N(t), V(t) \rangle \, dt,$$

in other words, $\frac{\partial}{\partial \varepsilon}\big|_{\varepsilon=0} L_\varepsilon$ is the L^2 inner product of κN and V, which means that the functional gradient of L_ε is κN. That is, the variation direction V resulting in fastest decreasing L_ε is $-N$ and the amount of decrease for a unit variation is κ.

Thus, an alternative approach to define the curvature normal is as the steepest descent variation minimizing the total length of the curve.

1.1.4 Turning Angle

Let us return to the simple definition of curvature

$$N' = \kappa T, \quad T' = -\kappa N$$

where N is the 90° clockwise rotation of T, *i.e.* N always points to the "right" if we face forward (sitting on the particle $\gamma(t)$). Evidently κ can take positive or negative values. When $\kappa > 0$ the particle moves towards the left; when $\kappa < 0$ the particle moves towards the right. Physically, κ is the angle the particle turns per unit arclength of movement.

To see this, let us consider $\mathbb{R}^2$ as $\mathbb{C}$, and express $\gamma = (\gamma_x, \gamma_y)$ as $\gamma_x + i\gamma_y$. Then the 90° clockwise rotation would be a multiplication by $-i$. Hence $N = -iT$. Since T has unit length, we write $T = e^{i\theta}$ where θ is the angle T makes with the positive real axis. Now, $T' = -\kappa N$ reads

$$(e^{i\theta})' = i\theta' e^{i\theta} = -\theta' N = -\kappa N$$

which shows

$$\theta' = \kappa.$$

A direct consequence is that the integral of κ is the total angle by which the direction of the particle has changed:

$$\Delta\theta = \int_{t_1}^{t_2} \kappa(t) \, dt.$$

In other words, we can just write down how much the tangent vector has turned:

$$T(t_2) = e^{i\Delta\theta} T(t_1) = e^{i \int_{t_1}^{t_2} \kappa(t) \, dt} T(t_1).$$

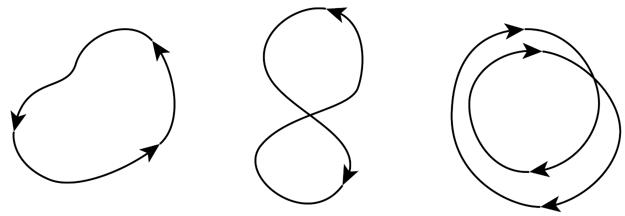


Figure 1.3: *Closed planar curves with total turning number* $+1$ *(left),* 0 *(middle) and* -2 *(right).*

1.1.5 Turning Number of Closed Curves

Suppose the curve γ is a closed arclength-parameterized regular curve. That is, $\gamma\colon[0,L]\to\mathbb{R}^2\cong\mathbb{C}$ with $\gamma(0)=\gamma(L)$, $\gamma'(0)=\gamma'(L)$, and $|\gamma'(t)|=1$ for all $t\in[0,L]$. By the definition of turning angle in the previous part we see

$$T(L)=e^{i\int_0^L\kappa(t)\,dt}\,T(0)$$

but $T(L)=T(0)$ from our assumption that γ is regular. Hence the **total turning angle**

$$\int_0^L\kappa(t)\,dt=2\pi m$$

must be an integer multiple of 2π. Here m is the **total turning number**, which is also the **winding number**, *i.e.*, the number of turns the tangent T makes around the origin.

Because the value of $\int_0^L\kappa(t)\,dt$ varies continuously under continuous variation of the function κ (say in the L^2 function space), and because $\int_0^L\kappa(t)\,dt$ takes on only quantized values, the total turning number m must stay constant when γ is "regularly deformed" (otherwise if m jumps, then $\int_0^L\kappa(t)$ does not vary continuously with respect to variation of κ). By regular deformation we mean the type of deformation of γ so that κ varies continuously in a suitable function space (say L^2). Such deformation is called a **regular homotopy**.

The total turning number m is a geometric quantity that is invariant under regular deformation of the geometry. That makes m a **topological quantity**.

So, if a regular closed curve is deformed under a regular homotopy, the total turning number stays invariant. Does the converse hold true? That is, given two closed regular curves having the same total turning number, does there exist a regular homotopy to deform one curve to the other? The answer is yes (**Whitney-Graustein Theorem**, whose proof is nicely explained in a classic video *Regular Homotopies in the Plane* filmed on a PDP-7 computer).

This is a big deal in topology. The difference in the turning number is the obstruction, and the *only* obstruction, preventing one closed curve to deform regularly to another. In other words, the turning number is a quantity that classifies all **regular homotopy types** (the set of different configurations that could not be connected by regular homotopic deformation) of regular closed curves.

1.2 Discrete Planar Curves

A discrete planar curve is a function $\gamma : \mathsf{I} \to \mathbb{R}^2$ where $\mathsf{I} = (0, 1, \ldots, n-1)$ is an ordered index set. Here $n \in \mathbb{N}$ is the number of points. The image $\gamma(\mathsf{I}) = (\gamma_0, \gamma_1, \ldots, \gamma_{n-1})$ is an ordered sequence of point positions in $\mathbb{R}^2$. Such a discrete curve is also called a **polygonal curve**. In **Houdini** it is called `"polyline"`. The vector $V_\mathsf{i} := \gamma_{\mathsf{i}+1} - \gamma_\mathsf{i}$ is called the edge vector, defined for $\mathsf{i} \in (0, 1, \ldots, n-2)$, the edge index set. The **tangent vector** is defined

$$T_\mathsf{i} := \frac{V_\mathsf{i}}{|V_\mathsf{i}|} = \frac{\gamma_{\mathsf{i}+1} - \gamma_\mathsf{i}}{|\gamma_{\mathsf{i}+1} - \gamma_\mathsf{i}|}, \quad \mathsf{i} \in (0, 1, \ldots, n-2).$$

Similar to the smooth case, we require the **regular** condition $\gamma_{\mathsf{i}+1} \neq \gamma_\mathsf{i}$ to have a well-defined tangent. With this tangent we define the **normal vector** N_i as the 90° clockwise rotation of T_i:

$$N_\mathsf{i} = -iT_\mathsf{i}.$$

Analogous to the smooth case, it would be convenient to reduce our considerations to arclength-parameterized curves.

Definition 1.6. A polygonal curve γ is **arclength-parameterized** if it has uniform edge lengths; that is, $|\gamma_{\mathsf{i}+1} - \gamma_\mathsf{i}| = \ell$ for some constant $\ell > 0$.

Given any discrete curve, finding its arclength re-parameterization might be tedious numerically. Luckily, **Houdini**'s Resample node can do it for us.

1.2.1 Discrete Curvature

How should one define curvature for a polygonal curve? Depending on which smooth approach we use to model our analogous discrete approach to follow, we will end up with different discrete curvatures.

Angles as curvature In the smooth theory, we first modeled curvature κ as the rate of change of the normal or tangent, *i.e.* $\kappa T = N'$, $-\kappa N = T'$ (with arclength parameterization). This approach can be brought to the discrete setting very directly: we can measure the discrete curvature by how much N changes. The tangent and normal change at vertices $\mathsf{i} \in (1,2,\ldots,n-2)$ (excluding the boundary vertices $0, n-1$) by an angle θ_{i}

$$T_{\mathsf{i}} = \exp(i\theta_{\mathsf{i}})T_{\mathsf{i}-1}, \quad N_{\mathsf{i}} = \exp(i\theta_{\mathsf{i}})N_{\mathsf{i}-1}.$$

That is, θ_{i} is the signed exterior angle at the polygonal vertex i.

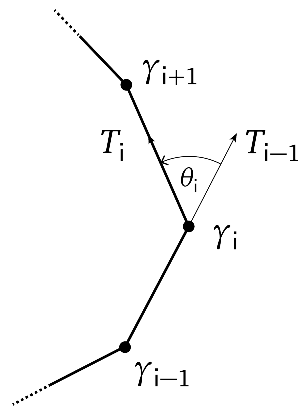


Figure 1.4: *Exterior angle of a vertex as a discrete curvature.*

Definition 1.7 (Angle based curvature)**.** The angle based discrete curvature is $\kappa^a_{\mathsf{i}} = \theta_{\mathsf{i}}$ for $\mathsf{i} \in (1,2,\ldots,n-2)$.

This discrete curvature, based on angles, places all curvature at vertices as Dirac distributions, leading to jumps θ_i. Hence the discrete curvature represents an integral quantity. The integral of curvature over some length of the curve then becomes the corresponding sum of individual curvatures at all contained vertices.

For a closed curve $\gamma : (0,1,\ldots,n-1,0) \to \mathbb{R}^2$, *i.e.* a closed polygon, the total sum of $\kappa^a_{\mathsf{i}} = \theta_{\mathsf{i}}$ is

$$\sum_{\mathsf{i}=0}^{n-1} \theta_{\mathsf{i}} = 2\pi m,$$

an integer multiple of 2π. Hence we see that the angle based curvature *preserves* the theory of turning number of closed curves.

Piecewise arc model An alternative perspective for modeling curvature is based on the osculating circle, where the curvature is the reciprocal of the radius. Let us consider this approach in the discrete setting.

At a vertex $\mathsf{i} \in (1,\ldots,n-2)$ of an arclength-parameterized polygonal curve $(\gamma_0,\ldots,\gamma_{n-1})$ we look at the two incident line segments $(\gamma_{\mathsf{i}-1},\gamma_{\mathsf{i}})$ and $(\gamma_{\mathsf{i}},\gamma_{\mathsf{i}+1})$. The two line segments, forming a wedge, determine a unique circle that is tangent to both line segments at their midpoints. It is not hard to see that the radius of this circle is $\frac{\ell}{2}\cot(\frac{\theta_{\mathsf{i}}}{2})$. See Figure 1.5.

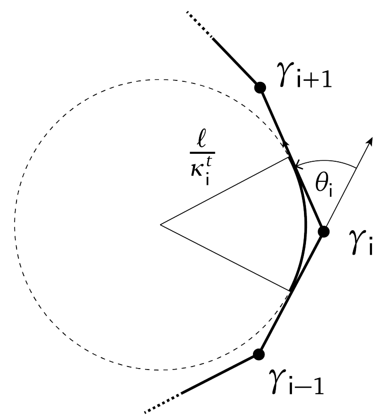


Figure 1.5: *The circle that is tangent to both line segments* $(\gamma_{\mathsf{i}-1}, \gamma_{\mathsf{i}})$ *and* $(\gamma_{\mathsf{i}}, \gamma_{\mathsf{i}+1})$ *at their midpoint.*

Trimming this circle, we draw an arc joining the midpoints $\frac{1}{2}(\gamma_{\mathsf{i}-1} + \gamma_{\mathsf{i}})$ and $\frac{1}{2}(\gamma_{\mathsf{i}} + \gamma_{\mathsf{i}+1})$. With such a construction at every $\mathsf{i} \in (1, \ldots, n-2)$, we obtain a continuous, piecewise arc, regular curve with continuous tangent. In other words, this construction *injects* the space of polygonal curve into a subset of the space of C^1 curves (the curves whose arclength parameterization is continuously differentiable, or in short C^1). In particular the constructed C^1 curves have piecewise constant curvature $\frac{2}{\ell}\tan(\frac{\theta_{\mathsf{i}}}{2})$, the reciprocal of the radius of each arc.

Definition 1.8. The discrete curvature based on the piecewise arc model is $\kappa^t_{\mathsf{i}} = 2\tan(\frac{\theta_{\mathsf{i}}}{2})$.

Note that we leave out the $\frac{1}{\ell}$ factor in the discrete curvature, treating it as an integrated quantity just as before with κ^a_{i}.

There is a nice property enjoyed by this tan(.)-involving discrete curvature. When one has a singular case $\theta_{\mathsf{i}} \to \pi$, whose continuous analogy is a *cusp* singularity, which has infinite curvature, one also has $\kappa^t_{\mathsf{i}} \to \infty$.

Blowing up of curvature plays a role in the elasticity model of curves. The elastic energy density of an elastic curve is proportional to $|\kappa|^2$. This implies that any curve with finite elastic energy $\int |\kappa|^2 < \infty$ is smooth without kink or cusp. Analogously a discrete curve with bounded $\sum |\kappa^t_{\mathsf{i}}|^2$ does not have any kink or cusp.

Variational approach Another approach to model curvature is by taking the variation of the total length. Suppose $\gamma^\varepsilon : (0, \ldots, n-1) \to \mathbb{R}^2$ is a one-parameter family of polygonal curves with γ_0 being arclength parameterized. The total length of the curve is given by

$$L^\varepsilon = \sum_{\mathrm{i}=1}^{n-1} |\gamma_\mathrm{i}^\varepsilon - \gamma_{\mathrm{i}-1}^\varepsilon|.$$

Let $V_\mathrm{i} = \frac{d\gamma_\mathrm{i}^\varepsilon}{d\varepsilon}\big|_{\varepsilon=0} \in \mathbb{R}^2$ be the variation of the curve. Like in the smooth case, we consider fixed boundary $V_0 = V_{n-1} = 0$. The variation of the total length at $\varepsilon = 0$ is

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} L^\varepsilon = \sum_{\mathrm{i}=0}^{n-1} \left\langle \frac{\partial L^\varepsilon}{\partial \gamma_\mathrm{i}^\varepsilon}, \frac{d\gamma_\mathrm{i}^\varepsilon}{d\varepsilon} \right\rangle_{\varepsilon=0} = \sum_{\mathrm{i}=1}^{n-2} \left\langle \frac{\partial L^\varepsilon}{\partial \gamma_\mathrm{i}^\varepsilon}\bigg|_{\varepsilon=0}, V_\mathrm{i} \right\rangle.$$

Since the only terms in L^ε involving $\gamma_\mathrm{i}^\varepsilon$ is $|\gamma_\mathrm{i}^\varepsilon - \gamma_{\mathrm{i}-1}^\varepsilon| + |\gamma_{\mathrm{i}+1}^\varepsilon - \gamma_\mathrm{i}^\varepsilon|$, we have

$$\begin{aligned}
\frac{\partial L^\varepsilon}{\partial \gamma_\mathrm{i}^\varepsilon}\bigg|_{\varepsilon=0} &= \frac{\partial |\gamma_\mathrm{i}^\varepsilon - \gamma_{\mathrm{i}-1}^\varepsilon|}{\partial \gamma_\mathrm{i}^\varepsilon}\bigg|_{\varepsilon=0} + \frac{\partial |\gamma_{\mathrm{i}+1}^\varepsilon - \gamma_\mathrm{i}^\varepsilon|}{\partial \gamma_\mathrm{i}^\varepsilon}\bigg|_{\varepsilon=0} \\
&= \frac{\gamma_\mathrm{i}^0 - \gamma_{\mathrm{i}-1}^0}{|\gamma_\mathrm{i}^0 - \gamma_{\mathrm{i}-1}^0|} - \frac{\gamma_{\mathrm{i}+1}^0 - \gamma_\mathrm{i}^0}{|\gamma_{\mathrm{i}+1}^0 - \gamma_\mathrm{i}^0|} \\
&= T_{\mathrm{i}-1} - T_\mathrm{i}.
\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} L^\varepsilon = \sum_{\mathrm{i}=1}^{n-2} - \langle T_\mathrm{i} - T_{\mathrm{i}-1}, V_\mathrm{i} \rangle,$$

which implies that the direction $T_\mathrm{i} - T_{\mathrm{i}-1}$ yields the steepest descent for the total length. One can check that

$$T_\mathrm{i} - T_{\mathrm{i}-1} = 2\sin\left(\frac{\theta_\mathrm{i}}{2}\right) \frac{N_{\mathrm{i}-1} + N_\mathrm{i}}{|N_{\mathrm{i}-1} + N_\mathrm{i}|}.$$

Definition 1.9. The discrete curvature based on the variational approach is $\kappa_\mathrm{i}^s = 2\sin(\frac{\theta_\mathrm{i}}{2})$.

1.2.2 Summary

Above we introduced three different discrete curvatures: $\kappa_\mathrm{i}^a = \theta_\mathrm{i}$, $\kappa_\mathrm{i}^t = 2\tan(\frac{\theta_\mathrm{i}}{2})$, and $\kappa_\mathrm{i}^s = 2\sin(\frac{\theta_\mathrm{i}}{2})$. From the point of view of approximation theory and truncation error analysis, all three are just a first order approximations of the definition $\kappa T = N'$. It is by understanding different geometric pictures for modeling curvature that one reveals the nature of different discrete curvatures. With a similar order of accuracy, one discrete curvature can have greater advantage over another depending on the application.

1.3 Space Curves

In this section we explore space curves in 3D space. Similar to the planar curves, given a parameter space $I = [a, b] \subset \mathbb{R}$, a **space curve** is a continuous function $\gamma : I \to \mathbb{R}^3$. For **regular space curves**, we also require γ to be differentiable and $\gamma' = \frac{d}{dt}\gamma \neq 0$ for all $t \in I$.

Example 1.2. The intersection curve of a cylinder $\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1\}$ and a surface $\{(x, y, z) \in \mathbb{R}^3 \,|\, z = x^2\}$ can be parameterized by a smooth function

$$\gamma(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ \frac{1+\cos(2t)}{2} \end{bmatrix}, \quad t \in [0, 2\pi),$$

with nowhere vanishing

$$|\gamma'(t)| = \sqrt{\sin^2(t) + \cos^2(t) + \sin^2(2t)}.$$

As before, the **tangent vector** of a smooth curve γ at $t \in I$ is defined by

$$T(t) := \frac{\gamma'(t)}{|\gamma'(t)|}.$$

A space curve γ is **arclength-parameterized** if $\left|\gamma'\right| = 1$. Similar to the construction for planar curves, all regular space curves can be re-parameterized by arclength.

Again, for an arclength-parameterized curve γ, we have $T = \gamma'$.

Example 1.3. The following family of space curves are parameterized by arclength.

$$\gamma(t) = \frac{1}{r + r^{2k-1}} \begin{bmatrix} r\cos(t) + \frac{1}{2k-1} r^{2k-1} \cos((2k-1)t) \\ r\sin(t) - \frac{1}{2k-1} r^{2k-1} \sin((2k-1)t) \\ \frac{2}{k} r^k \sin(kt) \end{bmatrix}$$

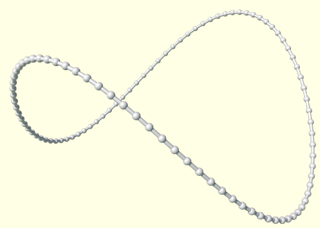
where $r > 0$ and $k \in \mathbb{N}$.

A **discrete space (polygonal) curve**, similar to a discrete planar curve, is an ordered sequence of $\mathbb{R}^3$ points $\gamma : \mathsf{I} = (0, 1, \ldots, n-1) \to \mathbb{R}^3$. The **discrete tangent vector** is $T_\mathsf{i} := (\gamma_{\mathsf{i}+1} - \gamma_\mathsf{i})/|\gamma_{\mathsf{i}+1} - \gamma_\mathsf{i}|$ for $\mathsf{i} \in (0, \ldots, n-2)$. An **discrete arclength-parameterized** curve has a constant $\ell = |\gamma_{\mathsf{i}+1} - \gamma_\mathsf{i}|$ for all $\mathsf{i} \in (0, \ldots, n-2)$.

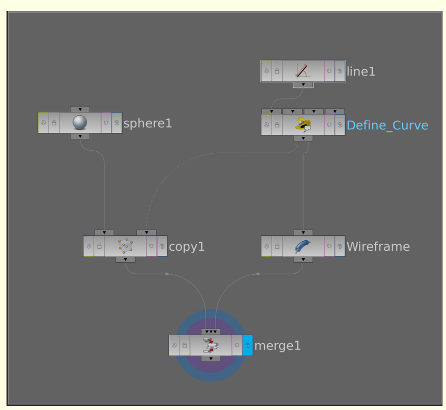
Exercise 1.2 (Visualize a space curve)**.** This is a practice for visualizing a space curve in **Houdini**. In a Geometry network, start with a Line node with Primitive Type "Polygon", Origin at $(0, 0, 0)$, Direction at $(1, 0, 0)$, Length equals 1, and Points equals, say, 50. From the Geometry Spreadsheet's point attributes () one sees that `v@P[x]` are 100 equally spaced numbers between 0.0 and 1.0. The left-most column shows the `i@ptnum` $\in \{0,\ldots,49\}$ (point number) attribute.

Pipe this node to a Point Wrangle node, and include the following code

```
#include "math.h" // for using PI
float t = @P.x * 2*PI;
@P.z = cos(t);
@P.x = sin(t);
@P.y = (1+cos(2*t))/2;
```



In order to visualize the points and edges of the curve properly, make use of the Copy and Wireframe nodes. The Copy node has two inputs; the first input (left) takes a geometry to be duplicated (for instance, a small sphere); the second input (right) would be a geometry whose set of points are the locations where the duplicated spheres will be placed. The Wireframe node replaces the edges of the input geometry by tubular surfaces with some thickness.



Are the vertices of this discrete space curve equally spaced (discrete arc-length parameterized)?

Here is an example how you can compute and visualize edge lengths. Create another Point Wrangle node after the earlier pointwrangle node for defining the curve. In it,

```
vector nextPosition = point(0,"P", i@ptnum + 1 );
f@edgeLength = length( nextPosition - v@P );
```

where the function `point( int` *input_node*`, string` *attrib_name*`, int` *point_number*`)` allows one to read the point attribute of an in-coming node *not necessarily the same point*. With the access to the "next point" (`i@ptnum + 1`) we can calculate the edge vector and its `length`, stored as a point attribute on the "source point" of the edge. As you may have expected, the last point (`if (i@ptnum == npoints(0)-1)`) will create a non-meaningful result, which we ignore in this practice. You can inspect the values of `f@edgeLength` from the Geometry Spreadsheet. Or you can put a Visualize node to display numbers as markers on the points.

Insert a Resample node after the pointwrangle node defining the curve. You can change the Measure to "Along arc" or "Along chord". Check that the "Along chord" option yields a discrete arc-length parameterization.

1.3.1 Curvature of Space Curves

So far there is no real difference between space curves and planar curves. The difference starts at the notion of normal vectors and curvature: N being the 90° "clockwise" rotation is meaningful only in the plane.

For a space curve, we call $Y(t) \in \mathbb{R}^3$ a **normal vector** if $\langle Y(t), T(t)\rangle = 0$.

In the 2D case, the curvature κ is defined to capture the rate of change of the tangent and normal. The sign of κ, in the 2D case, captures the turning orientation of the tangent with respect to the ambient orientation of the plane. In the 3D case, since there is no preferred orientation with respect to which we can describe the direction the curve is turning, we do not assign a sign to the curvature.

Definition 1.10. Let a space curve γ be parameterized by arclength. Then the **curvature** of the curve at $\gamma(t)$ is given by

$$\kappa(t) = |T'(t)|.$$

Recall that in the 2D case we have the formula $T' = -\kappa N$, which we can use in the 3D case to define an N.

Definition 1.11. The **curvature normal** of a arclength-parameterized space curve γ is given by

$$N(t) = -\frac{T'(t)}{|T'(t)|} = -\frac{T'(t)}{\kappa(t)}$$

provided that $\kappa(t) \neq 0$.

One checks that T' is always orthogonal to T by taking the derivative of $1 = \langle T, T\rangle$, which yields $0 = 2\langle T, T'\rangle$. That is, when $\kappa \neq 0$, N is a normal vector. The above notions of curvature and curvature normal create the following physical picture. The rate of change T' of the tangent vector T is the centripetal acceleration when moving along the curve at unit speed. Hence in for $-T' = \kappa N$ we have N pointing to the centrifugal direction with κ the magnitude of the acceleration.

Definition 1.12. For $\kappa \neq 0$ we define $B := T \times N$, called the **curvature binormal**.

In particular, (T, N, B) forms an orthonormal basis at every point on the curve γ whenever $\kappa \neq 0$.

From the local Taylor expansion of $\gamma(t)$ (assuming arclength-parameterization):

$$\begin{aligned}\gamma(t+\Delta t) &= \gamma(t) + \frac{\gamma'(t)}{1!}\Delta t + \frac{\gamma''(t)}{2!}\Delta t^2 + O\left(\Delta t^3\right)\\ &= \gamma(t) + T(t)\Delta t - \kappa(t)N(t)\frac{\Delta t^2}{2} + O(\Delta t^3)\end{aligned}$$

we see γ locally lies in the plane spanned by $T(t)$ and $N(t)$, called the **osculating plane** at $\gamma(t)$. The osculating plane contains the **osculating circle**, which is centered at $\gamma(t) - \frac{1}{\kappa(t)}N(t)$, normal to $B(t)$, and has radius $\frac{1}{\kappa(t)}$. The osculating circle is the best approximating circle describing local circular motion of $\gamma(t)$ (Figure 1.6).

Discrete Setting On an arclength-parameterized polygonal space curve $\gamma : (0, 1, \ldots, n-1) \to \mathbb{R}^3$, two consecutive tangents $T_{\mathsf{i}-1}, T_{\mathsf{i}}$ naturally span a 2D subspace $\mathrm{Span}\{T_{\mathsf{i}-1}, T_{\mathsf{i}}\}$ for $\mathsf{i} \in (1, \ldots, n-2)$ and for $T_{\mathsf{i}-1} \neq T_{\mathsf{i}}$ (analogous to the condition $\kappa \neq 0$). This 2D plane $\mathrm{Span}\{T_{\mathsf{i}-1}, T_{\mathsf{i}}\}$ is the discrete **osculating plane** of γ at i, as the neighboring segment of γ lies in this plane. Since the osculating plane is also characterized by the orthogonal complement of the curvature binormal, we define the discrete B_{i} be the normal of the osculating plane:

$$B_{\mathsf{i}} = -\frac{T_{\mathsf{i}-1} \times T_{\mathsf{i}}}{|T_{\mathsf{i}-1} \times T_{\mathsf{i}}|} = -\,\texttt{normalize}\,(T_{\mathsf{i}-1} \times T_{\mathsf{i}}).$$

On the osculating plane at i, the polygonal curve can be treated as a planar curve, and one may consider the discrete curvatures just as those for planar curves.

At this point one sees that T_{i} lives on edges whereas B_{i} lives at vertices. Where should N_{i} be defined? It can depend on the application. For example, consider N_{i} be defined on vertices,

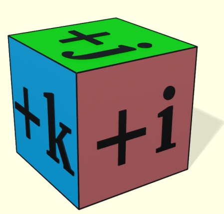
Figure 1.6: *Osculating circle.*

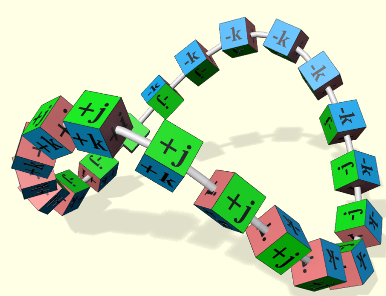
then a natural construction would be

$$N_i = \texttt{normalize}\left(B_i \times \texttt{normalize}\left(T_{i-1} + T_i\right)\right)$$

which is orthogonal to both B_i and the interpolated T.

Exercise 1.3 (Copy with orientation). In Exercise 1.2, we learned how to use Copy node to create copies of geometries located on the points, which allows us to visualize the vertices of a polygonal curve. If we can further control the 3D orientation of these copies, we can visualize, for instance, (T, N, B). To specify the orientation of the geometry to be copied, define a point attribute "`p@orient`" (`p@` is a **`vector4`** type, *i.e.* quaternion) for the points of the curve. Copy will find `p@orient` to pre-rotate the duplicated geometry. Copy will also find a **`float`** attribute named `f@pscale` to scale the duplicated geometry.





In this figure, the quaternion `p@orient` has been carefully chosen so that it rotates the standard basis $(\mathbf{i},\mathbf{j},\mathbf{k})$ to (T,N,B). How do we assign the values for `p@orient` properly?

Recall that a quaternion $q \in \mathbb{H}$, $|q| = 1$, represents a 3D rotation $\mathbf{y} \in \mathbb{R}^3 \mapsto q\mathbf{y}\bar{q} \in \mathbb{R}^3$, which rotates $\mathbf{y}$ an angle θ about an axis $\mathbf{v} \in \mathbb{R}^3$, $|\mathbf{v}| = 1$, when $q = \cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2})\mathbf{v}$. In VEX language, $q =$ `quaternion`$(\theta, \mathbf{v})$; alternatively,

$$q = \texttt{set}\left(\sin\left(\tfrac{\theta}{2}\right)\mathbf{v}.\texttt{x}, \sin\left(\tfrac{\theta}{2}\right)\mathbf{v}.\texttt{y}, \sin\left(\tfrac{\theta}{2}\right)\mathbf{v}.\texttt{z}, \cos\left(\tfrac{\theta}{2}\right)\right),$$

where `set` is the function to produce `vector2`, `vector`, or `vector4` types. Note that the real part of q sits at the last component of the 4-vector in VEX. To multiply quaternions q_1q_2, use `qmultiply`(q_1, q_2). To rotate a vector $\mathbf{y} \in \mathbb{R}^3$ by a quaternion $q \in \mathbb{H}$, `qrotate`$(q, \mathbf{y})$.

There is a handy VEX function `vector4 dihedral(vector x, vector y)`, defined as

$$\texttt{dihedral}(\mathbf{x}, \mathbf{y}) := \texttt{quaternion}(\theta, \mathbf{v})$$

where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$, and $\mathbf{v} =$ `normalize`$(\mathbf{x} \times \mathbf{y})$. You can understand `dihedral`$(\mathbf{x}, \mathbf{y})$ as the shortest rotation mapping $\mathbf{x}$ to $\mathbf{y}$.

Now, suppose at every point i we have a (positively-oriented) orthonormal basis $(E_\mathrm{i}, F_\mathrm{i}, G_\mathrm{i})$ defined (for instance $(T,N,B)_\mathrm{i}$ with interpolated T). We want to find a quaternion $q_\mathrm{i} \in \mathbb{H}$ so that it rotates the standard basis $(\mathbf{i},\mathbf{j},\mathbf{k})$ to $(E_\mathrm{i}, F_\mathrm{i}, G_\mathrm{i})$. We can construct q by $q = q_2q_1$, where $q_1 =$ `dihedral`$(\mathbf{i}, E)$ rotates $\mathbf{i}$ to E, and $q_2 =$ `dihedral`$(q_1\mathbf{j}\bar{q}_1, F)$ rotates $q_1\mathbf{j}\bar{q}_1$ to F.

```
// assume v@E, v@F, v@G has been defined and are orthonormal
vector i = set(1,0,0);
vector j = set(0,1,0);
vector k = set(0,0,1);

vector4 q1 = dihedral( i, v@E );
vector4 q2 = dihedral( qrotate(q1,j), v@F );
vector4 q = qmultiply(q2,q1);

p@orient = q;
```

Can you see why this works?

1.3.2 Parallel Frame for Discrete Curves

In Exercise 1.3 we visualize an orthonormal basis with a quaternion `p@orient` $= q \in \mathbb{H}$ at each point of γ. This leads to the notion of **frame**. Specifically, a frame on a curve γ is an orthogonal basis (ONB) (T, N_1, N_2) aligned with its tangent. Here N_1, N_2 do not need to be the curvature normal N and curvature binormal B. The frame with $N_1 = N$ is known as the **Frenet**

frame, which we discuss in more detail in the next section. Another special frame we would be interested in is a **parallel frame**, where (T, N_1, N_2) is chosen so that, as it varies along γ, it has a minimal amount of "twist".

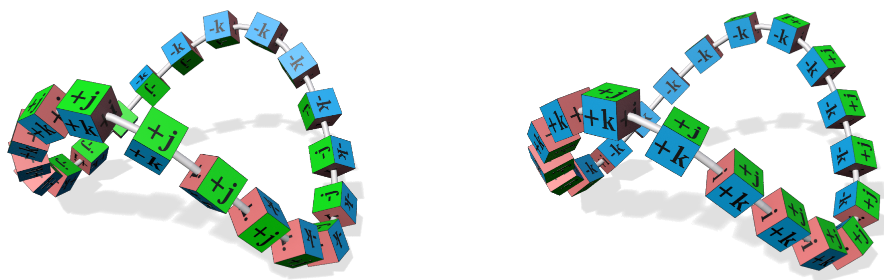


Figure 1.7: *The Frenet frame (left) and a parallel frame (right).*

We will discuss "twist" for a framed curve in the smooth setting in the next section. Before that let us talk about parallel frames in the discrete setup. An example of discrete parallel frame is a chain of interlinked rings as shown in Figure 1.8. At each point $\mathrm{i} \in (0, \ldots, n-1)$ let there be a tangent vector T_i. You can choose it to be the tangent we introduced $T_\mathrm{i} = \texttt{normalize}(\gamma_{\mathrm{i}-1} - \gamma_\mathrm{i})$ or an interpolated tangent $\tilde{T}_\mathrm{i} = \texttt{normalize}(T_{\mathrm{i}-1} + T_\mathrm{i})$. These are the "direction" of the metal ring in the chain. Let $q_\mathrm{i} \in \mathbb{H}$ be the 3D orientation of the metal ring. For that being said,

$$q_\mathrm{i} \mathbf{i} \bar{q}_\mathrm{i} = T_\mathrm{i}.$$

The physical condition for the chain is that, at the point where the ring interlocks with the next ring, the two orientations are "hinged": $q_{\mathrm{i}+1}$ is q_i additionally rotated by

$$\begin{aligned} q_{\mathrm{i}+1} q_\mathrm{i}^{-1} &= \texttt{dihedral}(T_\mathrm{i}, T_{\mathrm{i}+1}) \\ &= \cos\left(\tfrac{\theta_\mathrm{i}}{2}\right) + \sin\left(\tfrac{\theta_\mathrm{i}}{2}\right)(-B_{\mathrm{i}+1}) \end{aligned}$$

where the rotation angle θ_i is the angle from T_i to $T_{\mathrm{i}+1}$, and the rotation axis $B_{\mathrm{i}+1} = -\,\texttt{normalize}(T_\mathrm{i} \times T_{\mathrm{i}+1})$ is normal to the plane spanned by T_i and $T_{\mathrm{i}+1}$ (osculating plane). The recurrence relation

$$q_{\mathrm{i}+1} = \texttt{dihedral}(T_\mathrm{i}, T_{\mathrm{i}+1}) q_\mathrm{i}$$

is called the **parallel transport**. By propagating q_i along the entire curve, we obtain a **parallel frame**. A general frame, for comparison, results when $q_{\mathrm{i}+1}$ is *first* parallel transported from q_i *and then* additionally rotated about $T_{\mathrm{i}+1}$ with an angle we call **twist**.

Exercise 1.4. Build a parallel frame along a curve. First compute the tangent T_i. Define a point attribute `p@orient` later to be assigned. To add a point attribute called `p@orient` you can use a Point Wrangle running over points, with a single line of code

```
p@orient; // works only when running over points
```

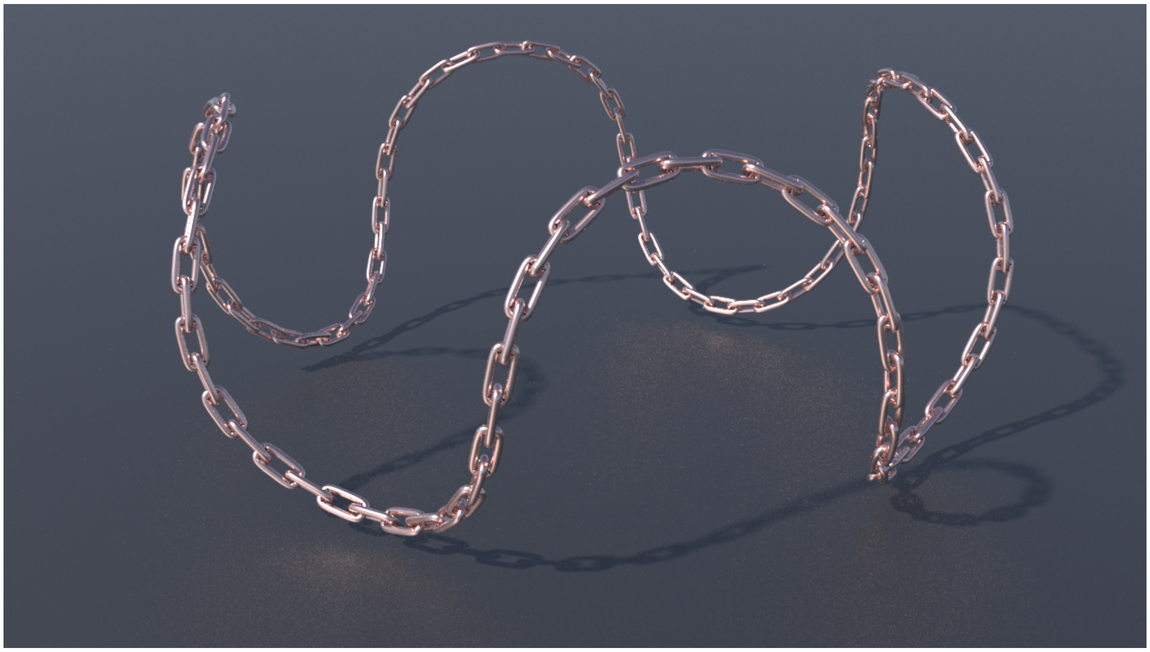


Figure 1.8: *The metal rings are copied along the curve according to a parallel frame (with additional alternating* 90° *rotation about the tangent).*

or

```
    addpointattrib( geoself(), "orient", {0,0,0,1} );
    // this works also for running over vertice, primitives or detail once.
```

Now, we want to define, at $\mathsf{i} = 0$, `p@orient` $= q_0 \in \mathbb{H}$ arbitrarily so that $q_0 \mathbf{i} \bar{q}_0 = T_0$. And then we iteratively define

$$q_{\mathsf{i}+1} = \texttt{dihedral}(T_{\mathsf{i}}, T_{\mathsf{i}+1})(q_{\mathsf{i}}).$$

This requires a loop (it is not possible to parallelize the computation at each point). To do so, drop an Attribute Wrangle, and set "Run Over" to "Detail (only once)". In that case, unlike running over points, the code is executed on the entire geometry once instead of each individual point in parallel.

Fill in the following code.

```
int n = npoints(0); // number of points of the 0-th input node
vector4 q;

// TODO: define your initial q

// set orient attribute at ptnum==0 to be q
setpointattrib( geoself(), "orient", 0, q );

// for loop
for (int i=0; i<n-1; i++){
    // TODO: update your q here

    // set orient attribute at ptnum==i+1 to be q
    setpointattrib( geoself(), "orient", i+1, q );
}
```

You can also use a `while` or `do-while` loop with the same syntax as C.

Visualize the parallel frame by applying to a Copy node to generate Figure 1.8 or a similar visualization.

1.4 Framed Curves: Smooth Theory

A **frame** on a curve $\gamma : I \to \mathbb{R}^3$ is a position-dependent 3D ONB $(e_1, e_2, e_3)(t)$, $\langle e_i, e_j\rangle = \delta_{ij}$, at every point along the curve. A natural choice of frame should reflect geometric properties of the curve. Thus in almost all cases one considers $e_1(t) = T(t)$. The rest of the axes e_2, e_3 span the **normal space** of the curve, and hence we later denote e_2, e_3 by N_1, N_2. We also require the frame be positively oriented, that is, $N_2 = T \times N_1$.

For a frame (T, N_1, N_2), any linear combination $\alpha N_1 + \beta N_2$ is a **normal vector** of the curve.

Throughout the section, we assume γ is parameterized by arclength.

Let us first study the classic – the Frenet frame.

1.4.1 Frenet Frame

Frenet frame is a particular choice of frame (T, N_1, N_2) with $N_1 = N$ the curvature normal

$$N(t) = -\frac{T'(t)}{|T'(t)|}, \quad T'(t) = -\kappa(t)N(t)$$

and $N_2 = B = T \times N$ the curvature binormal. For simplicity in the context of Frenet frame we assume $\kappa = |T'| \neq 0$.

Recall that the osculating plane $\mathrm{Span}\{T,N\}$ is the plane in which γ lies up to second order. If the curvature binormal stays constant, then the space curve γ lies in the osculating plane entirely. The **torsion** measures the rate of change of the curvature binormal.

Lemma 1.2. *The derivative B' is always parallel to N.*

Proof. Taking derivative of $1 = \langle B, B\rangle$ yields $0 = 2\langle B', B\rangle$, and taking derivative of $0 = \langle T, B\rangle$ yields $0 = -\kappa\langle N, B\rangle + \langle T, B'\rangle = \langle T, B'\rangle$. Thus B' is perpendicular to T and B, so B' is parallel to N. □

Definition 1.13 (Torsion)**.** Let γ be an arclength-parameterized space curve with non-vanishing curvature. The **torsion** $\tau(t)$ of the curve is given by the coefficient

$$B'(t) = -\tau N(t).$$

Lemma 1.3. $N' = \kappa T + \tau B$.

Proof. Since N is unit, $\langle N', N\rangle = 0$ (by taking derivative of $\langle N, N\rangle = 1$), *i.e.*, N lies in the span of B and T. The derivative of $0 = \langle N, T\rangle$ gives $0 = \langle N', T\rangle + \langle N, T'\rangle = \langle N', T\rangle - \kappa$. Hence $\langle N', T\rangle = \kappa$. On the other hand, the derivative of $0 = \langle N, B\rangle$ is $0 = \langle N', B\rangle + \langle N, B'\rangle = \langle N', B\rangle - \tau$, and thus $\langle N', B\rangle = \tau$. Therefore $N' = \kappa T + \tau B$. □

The relation $T' = -\kappa N$, $N' = \kappa T + \tau B$, and $B' = -\tau N$ are summarized as the Frenet-Serret formula.

Theorem 1.1 (Frenet-Serret Formula)**.** *For an arclength-parameterized curve, the Frenet frame (T, N, B) satisfies*

$$\frac{d}{dt}\begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix} = \begin{bmatrix} & -\kappa(t) & \\ \kappa(t) & & \tau(t) \\ & -\tau(t) & \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}. \tag{1.1}$$

Note that the Frenet-Serret formula is a system of ordinary differential equation of the form $Y' = AY$ with an skew-symmetric system matrix $A^\intercal = -A$. A direct consequence of A being skew-symmetric is that $(Y^\intercal Y)' = 0$. Hence, if initially (T, N, B) is orthonormal, *i.e.* $YY^\intercal = Y^\intercal Y = I$, the solution to Eq. (1.1) is guaranteed to be orthonormal.

Corollary 1.1 (Fundamental Theorem of Space Curve)**.** *Every smooth space curve with non-zero curvature is determined by its curvature and torsion up to a global rotation and translation.*

Proof. Given the pair of scalar functions $\kappa(t)$ and $\tau(t)$, say for $t \in [0, L]$, Eq. (1.1) is uniquely integrated into a solution (T,N,B) up to a choice of initial orientation $(T,N,B)|_{t=0}$. Then one integrates $\gamma(t) = \gamma(0) + \int_0^t T(\tilde{t})\,d\tilde{t}$, determining an arclength-parameterized $\gamma(t)$ uniquely up to a choice of translation $\gamma(0)$. □

Example 1.4 (Aerobatics)**.** A simple model for airplane flight is that the airplane, whose position is described by a curve $\gamma : [0,T] \to \mathbb{R}^3$, has a constant speed, say $|\gamma'| = 1$. The typical steer control of an aircraft is on the pitch and the roll, ignoring the yaw; they correspond to rotating the aircraft about the axis pointing to the left-right, front-back and up-down direction respectively in pilot's point of view. In such a model, the pilot's frame is the Frenet frame: T points to pilot's front, N to pilot's up (or down) and $B = T \times N$ to pilot's left or right. The pilot controls $\kappa(t)$ and $\tau(t)$ as the rate of change of the pitch and roll.

Figure 1.9: *An aircraft controlled based on pitch and roll follows the Frenet frame.*

1.4.2 General Frame

An obvious problem with the curvature normal vector N is that it is not well-defined when $\kappa = 0$. This brings us to a more modern approach for modeling the normals and curvature. In this setup, consider a normal vector field N_1 arbitrarily; *i.e.* let $N_1 : I \to \mathbb{R}^3$, $|N_1(t)| = 1$ and $N_1(t) \perp T(t)$ for all $t \in I$. This N_1 induces $N_2 := T \times N_1$, so (T, N_1, N_2) is some orthonormal basis for $\mathbb{R}^3$ varying along the curve. We can define $\kappa_1 = \langle N_1', T\rangle$ similar to the planar curve. Similarly $\kappa_2 = \langle N_2', T\rangle$. We put κ_1 and κ_2 together and write a **complex curvature**

$$\psi = \kappa_1 + i\kappa_2.$$

Indeed κ_1, κ_2 are some notions of curvatures that measures the rate of change of N_1 and N_2, but both of them depend on the choice of N_1 at the first place. It turns out that, if we make a different choice of N_1, say $N_1 \mapsto \tilde{N}_1 =$ "N_1 rotated about T by angle q", then $\psi \mapsto \tilde{\psi} = e^{-iq}\psi$. This is called a **gauge transformation**. Quantities written in terms of ψ which are **gauge invariant** would be geometrically meaningful for the curve (independent of the choice of basis). For example the norm $|\psi|$ is gauge invariant, since $|e^{-iq}\psi|$ is unchanged for any q. In fact $\kappa = |\psi|$. There are other measurements, such as the **twist** $\omega = \langle N_1', N_2 \rangle$ as a quality of the frame (T, N_1, N_2), that depend on the choice of N_1.

In application, N_1, N_2 can describe the guide line for a solid material thickened around the curve, the metal rings of a chain (Figure 1.8), or a model for DNA molecules.

Complex representation

Given a frame (T, N_1, N_2), every normal vector $Y \in \mathbb{R}^3$, $Y \perp T$, can be written as $Y = aN_1 + bN_2$ for some unique coefficients $a, b \in \mathbb{R}$. Conveniently, we represent the coefficients a, b by a complex number $z = a + bi \in \mathbb{C}$, and hereby write

$$Y = (a + bi)N_1 = zN_1.$$

This amounts to overloading the scalar multiplication of normal vectors by complex numbers: for each $Y \in \operatorname{Span}\{N_1, N_2\}$, define $iY = T \times Y$. One should check that we indeed have $T \times (T \times Y) = -Y$, agreeing with the fact that $i^2 = -1$; this ensures that there is no violation of the associativity of scalar multiplications $z_1(z_2Y) = (z_1z_2)Y$. Now, we have in particular $N_2 = iN_1$, and thus $aN_1 + bN_2 = zN_1$ for $z = a + bi$.

Recall that in the exercises we did in the discrete setting, the frame is represented via a quaternion $q \in \mathbb{H}$, $|q| = 1$, with $q\mathbf{i}\bar{q} = T$, $q\mathbf{j}\bar{q} = N_1$ and $q\mathbf{k}\bar{q} = N_2$. A cute thing about the complex representation here is that, for $Y = (a + bi)N_1$ we have

$$Y = (a + bi)\ \texttt{qrotate}(q, \mathbf{j}) = \texttt{qrotate}(q(a + b\mathbf{i}), \mathbf{j}).$$

Note Mathematically, an operator such as "$T\times$", which acts on the vector space $\operatorname{Span}\{N_1, N_2\}$ satisfying $(T\times) \circ (T\times) = -\mathrm{id}$, is called a **complex structure** for the vector space.

Definition 1.14 (Complex curvature). For an arclength-parameterized space curve $\gamma : I \to \mathbb{R}^3$ with a given frame $(T, N_1, N_2)(s)$, $s \in I$, the derivative $T'(s) = -\kappa(s)N(s) \in \operatorname{Span}\{N_1(s), N_2(s)\}$ is written as

$$T'(s) = -\kappa(s)N(s) = -\psi(s)N_1(s)$$

for some $\psi(s) \in \mathbb{C}$. The complex-valued function $\psi : I \to \mathbb{C}$ is the **complex curvature** of the curve γ under the frame (T, N_1, N_2).

Recall the curvature of the curve $\kappa(s) = |T'(s)|$, and thus $\kappa(s) = |\psi(s)|$ the absolute value of the complex curvature.

Proposition 1.1. $\kappa = |\psi|$.

While the derivative of T yields the curvature, the derivative of N_1 and N_2 gives information on how much the frame is twisted. First, note that $N_1' \in \mathrm{Span}\{T, N_2\}$ and $N_2' \in \mathrm{Span}\{T, N_1\}$. The component of N_1', N_2' along T can be calculated by taking derivatives of $0 = \langle N_1, T\rangle = \langle N_2, T\rangle$:

$$\langle N_1', T\rangle = \mathrm{Re}(\psi), \quad \langle N_2', T\rangle = \mathrm{Im}(\psi).$$

On the other hand, by taking derivative of $\langle N_1, N_2\rangle = 0$ one obtains

$$\langle N_1', N_2\rangle = -\langle N_1, N_2'\rangle.$$

Definition 1.15 (Twist)**.** The **twist** of the frame (T, N_1, N_2) is defined by

$$\omega := \langle N_1', N_2\rangle = -\langle N_1, N_2'\rangle.$$

Corollary 1.2. *A general frame* (T, N_1, N_2) *satisfies*

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} & -\mathrm{Re}(\psi) & -\mathrm{Im}(\psi) \\ \mathrm{Re}(\psi) & & \omega \\ \mathrm{Im}(\psi) & -\omega & \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}.$$

Note that $(T, N_1, N_2) = (T, N, B)$ is the Frenet frame if and only if one has $\psi \in \mathbb{R}$, $\psi = \kappa$. And the torsion τ is the twist of the Frenet frame.

Gauge Transformation

Now let us look at how does ψ and ω transform as we change a frame. Suppose $(T, \tilde{N}_1, \tilde{N}_2)$ is another frame with

$$\tilde{N}_1 = e^{i\alpha} N_1$$

for some $\alpha \in \mathbb{R}$. Suppose $\tilde{\psi}$ and $\tilde{\omega}$ are the complex curvature and the twist of the new frame $(T, \tilde{N}_1, \tilde{N}_2)$. Then we have $T' = \psi N_1 = \psi e^{-i\alpha} e^{i\alpha} N_1 = e^{-i\alpha}\psi \tilde{N}_1$, which shows that

$$\tilde{\psi} = e^{-i\alpha}\psi.$$

The twist of the new frame $\tilde{\omega} = \langle \tilde{N}_1', \tilde{N}_2\rangle = \langle (e^{i\alpha}N_1)', e^{i\alpha}N_2\rangle = \langle i\alpha' e^{i\alpha}N_1, ie^{i\alpha}N_1\rangle + \langle e^{i\alpha}N_1', e^{i\alpha}N_2\rangle$, which gives

$$\tilde{\omega} = \omega + \alpha'.$$

Parallel Frame

Definition 1.16. A frame (T, N_1, N_2) is called a **parallel frame** if it has no twist ($\omega = 0$).

For a non-closed curve $\gamma : [0, L] \to \mathbb{R}^3$ (as oppose to closed curve which has its domain $[0, L]$ replaced by a periodic domain), one can always obtain a parallel frame.

Suppose (T, N_1, N_2) is an existing frame which is not necessarily parallel, *i.e.* $\omega \neq 0$. We can "untwist" the frame, $\tilde{N}_1(s) = e^{i\alpha(s)} N_1(s)$, with $\alpha(s)$ designed by the following. From the gauge transformation rule $\tilde{\omega} = \omega + \alpha'$ we see that $(T, \tilde{N}_1, \tilde{N}_2)$ will be parallel if $\alpha' = -\omega$, *i.e.* $\alpha(s) = \alpha(0) - \int_0^s \omega(t)\, dt$. Note that in general $\alpha(L) \neq \alpha(0)$, so we may not obtain a parallel frame on a closed curve without discontinuity in its frame.

Another way to generate a parallel frame is by solving

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} & -\mathrm{Re}(\psi_P) & -\mathrm{Im}(\psi_P) \\ \mathrm{Re}(\psi_P) & & \\ \mathrm{Im}(\psi_P) & & \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix},$$

which is the **parallel transport** we did in the discrete setting.

Torsion in terms of a Parallel Frame

Now, suppose ψ_P is the complex curvature of a parallel frame (T, N_1, N_2). Let us write ψ_P in polar form $\psi_P = \kappa e^{i\theta}$. If we apply a gauge transformation $\tilde{N}_1 = e^{i\theta} N_1$, which implies $\tilde{\psi} = e^{-i\theta}\psi_P = \kappa$, and thus that $(T, \tilde{N}_1, \tilde{N}_2)$ is the Frenet frame. The parallel frame (T, N_1, N_2) has no twist, so the transformed frame, now the Frenet frame (T, N, B), has a twist $\tilde{\omega} = 0 + \theta' = \tau$, since the torsion is the twist of the Frenet frame.

Proposition 1.2. *If $\psi_P = \kappa e^{i\theta}$ is the complex curvature with respect to a parallel frame, then the torsion $\tau = \theta'$. In other words, $\psi_P(s) = \kappa(s) e^{i\theta(0)} e^{i\int_0^s \tau}$.*