# Chapter 3

# Surface Curvatures

In the chapter on curves we discussed tangent and curvature of a unit speed traversed curve. Here we want to apply the ideas from curves to the surface setting. This will lead us eventually to a number of variational characterizations of curvatures and related quantities which will turn out to be very useful in practice.

Suppose our surface is parameterized by a function $f : \mathbb{R}^2 \supset M \to \mathbb{R}^3$ for $M$ some suitable parameter domain (or abstract manifold). The first thing we want to do is measure lengths and areas on the surface. For this we introduce the **metric** $g : T_pM \times T_pM \to \mathbb{R}$. It depends smoothly on $p \in M$ and plays the role of the inner product in each tangent space as a function of the parameterization. It is also known as the **first fundamental form**. Since we have an embedding in $\mathbb{R}^3$ a natural choice for the metric is the inner product in $\mathbb{R}^3$

$$g(X,Y) := \langle df(X), df(Y)\rangle$$

where $X, Y$ are two tangent vectors. In other words, $g(.,.)$ tells us about lengths of tangent vectors to the surface in terms of tangent vectors from the domain. We also say that the metric $g$ here is induced by the parameterization $f$.

Next we define the normal $N$ to the surface as the (unique up to sign) unit length vector which satisfies

$$\langle df(X), N\rangle = 0$$

for all tangent vectors $X$. For an **orientable** smooth manifold we can in fact make a global choice so that $N$ varies smoothly.

In practice we construct the normal as the cross product of partial derivatives of the parameterization

$$N = \frac{df(X) \times df(Y)}{|df(X) \times df(Y)|}$$

where the norm is defined in the usual way $|df(X)| = \langle df(X), df(X)\rangle^{1/2}$. Here we assume, as we always do, that $f$ is smooth and regular, *i.e.*, $df$ has full rank. A natural choice for $X$ and $Y$ would be two orthonormal basis vectors from the domain. We may now think of $N\colon M \to \mathbb{S}^2 \subset \mathbb{R}^3$ as another geometry, the so-called **Gauss map** associated with the domain.

Having a metric we can define lengths, angles, and measure areas on the surface as functions of the domain. For example, the area of a surface can be found by integrating the 2-form

$$A_M = \int_M \tfrac{1}{2}\langle N, df \wedge df\rangle = \int_M \det(N, df, df)$$

over the parameter domain.

*Remark* 3.1. What is the meaning of $df \wedge df$ in the expression for the area? This is the first instance of a 1-form which is vector valued so to have a meaningful wedge product for such 1-forms we need a multiplication operation for the $\wedge$. One such multiplication operation that works (and the one we intend here) is the cross product. Letting $\alpha, \beta \in \Omega^1(M;\mathbb{R}^3)$ and $X, Y \in T_pM$ we define

$$\alpha \wedge \beta(X,Y) := \alpha(X) \times \beta(Y) - \alpha(Y) \times \beta(X).$$

The resulting 2-form is obviously bi-linear and alternating. In particular $df \wedge df = 2df \times df$ because the cross product as a multiplication operation is already alternating. So in particular the rule from the $\mathbb{R}$-valued form setting that $\alpha \wedge \alpha = 0$ for $\alpha \in \Omega^1(M;\mathbb{R})$ does not apply here. Buyer beware!

Ok. That was easy. More interesting is the curvature (“bending”) of the surface. In the curve case we got a handle on curvature by looking at the change in tangent or normal. Because the two are related by a $\frac{\pi}{2}$ rotation they were equivalent. In the case of a 2-manifold orientable surface there is no distinguished tangent vector so we look at the change in normal vector. This is measured by the differential of $N$, the so-called **Weingarten map**, $dN\colon T_pM \to T_{N(p)}\mathbb{S}^2 \triangleq df(T_pM)$. Notice that we pulled a trick here. The Gauss map $N\colon M \to \mathbb{S}^2$ goes from the parameter domain to the 2-sphere. Its differential therefore goes to the tangent space of the 2-sphere. But the tangent space to the surface at some point $p$ and the tangent space to the Gauss map at $p$ are parallel since they share the same normal. Hence we can identify them and treat $dN$ as a mapping of $T_pM$ to the tangent space of the surface, $df(T_pM)$. As a consequence there exists a linear operator $A\colon T_pM \to T_pM$ such that $dN = df \circ A$. This linear operator is called the **shape operator**. Putting it all together we can define the **second fundamental form**

$$II_p(X,Y) := \langle dN(X), df(Y)\rangle = g(AX,Y).$$

Surprisingly $A$ is self adjoint with respect to $g$, *i.e.*, $g(AX,Y) = g(X,AY)$. To see this note that $\langle df, N\rangle = 0$ and thus its derivative vanishes as well

$$0 = d\langle df, N\rangle = \langle df \wedge dN\rangle.$$

Substituting $X, Y$ and expanding the definition we get

$$\langle df \wedge dN\rangle(X,Y) := \langle df(X), dN(Y)\rangle - \langle df(Y), dN(X)\rangle.$$

Setting the latter equal to zero the selfadjointness of $A$ follows.

*Remark* 3.2. Whoa! What happened here with the symbol $\langle df \wedge dN\rangle$? Again we are dealing with a wedge product on vector valued 1-forms. Above we used the cross product as our multiplication operation. Here the multiplication operation is the inner product $\langle .,. \rangle$, which is what the notation of $\langle \ldots \rangle$ surrounding the wedge product of vector valued 1-forms indicates. As before we see that 2-form thus defined is bi-linear and alternating as it should be. Notice that in this case we get $\langle \alpha \wedge \alpha\rangle = 0$ for any $\alpha \in \Omega^1(M;\mathbb{R}^3)$.

We can now define the **normal curvature** of the surface in the direction $X$ at $p$ as

$$\kappa^N(X) := \frac{\langle dN(X), df(X)\rangle}{\langle df(X), df(X)\rangle} = \frac{g(AX,X)}{g(X,X)},$$

*i.e.*, the variation of the surface normal $N$ in the direction $X$ normalized by whatever stretching the parameterization introduced. Hence $\kappa^N(X)$ gives us the curvature of the surface in the tangent direction $X$. Had we considered a curve $\gamma$ within the surface, with $\gamma(0) = p$ and $T(0)$ co-linear with $df(X)$ our usual calculation for the curvature of $\gamma$ at $p$ would have given us

$$\kappa^\gamma = \langle N'_\gamma, T_\gamma\rangle.$$

This expression formally coincides with the expression for $\kappa^N(X)$. It projects the differential of $N_\gamma$ onto the given tangent direction. No normalization is needed since we are assuming an arc length parameterization. There is one significant difference though. Instead of the surface normal $N$ it uses the curve normal $N_\gamma$, giving us the relationship

$$\kappa^N(X) = \kappa^\gamma \langle N, N_\gamma\rangle.$$

In fact we can split the curvature $\kappa^\gamma$ of a given curve on the surface into a normal component $\kappa^N$ and a tangential component $\kappa^T$. The normal component keeps the particle on the surface (constraint force) while the particle experiences lateral acceleration proportional to the tangential component of the curvature. The latter is referred to as the **geodesic curvature** and will become relevant when we talk about, you guessed it, geodesics. Such curves on a surfaces are characterized by having zero geodesic curvature making them "straightest" on a curved surface. But more on that later.

Earlier we observed that $dN = df \circ A$ where $A$ is the shape operator, a self adjoint linear operator from $\mathbb{R}^2$ to $\mathbb{R}^2$. Because it is self adjoint it diagonalizes with respect to an orthogonal basis of eigenvectors. Let these be $\{X, Y\}$, forming a positively oriented ONB for the tangent space and call the eigenvalues $\kappa_1$ resp. $\kappa_2$

$$dN(X) = \kappa_1\, df(X) \qquad \text{resp.} \qquad dN(Y) = \kappa_2\, df(Y).$$

The eigenvectors are referred to as **principal curvature directions** while the eigenvalues $\kappa_1$ and $\kappa_2$ give the maximal resp. minimal curvatures.

Since $A$ is a linear operator from $\mathbb{R}^2$ to $\mathbb{R}^2$ it has two invariants which are independent of the basis chosen. The determinant

$$\det(A) = \kappa_1 \kappa_2 =: K$$

which is called the **Gaussian curvature** and the trace

$$\mathrm{tr}(A) = \kappa_1 + \kappa_2 =: 2H$$

called the **mean curvature** up to the factor of 2, which some authors forgo.

**Exercise 3.1.** Show that the mean curvature is indeed the mean curvature over all directions in the tangent space

$$H = \frac{1}{2\pi} \int_0^{2\pi} \kappa^N(X^\theta)\, d\theta$$

where $\theta \in [0, 2\pi]$ parameterizes the unit circle of directions $X^\theta$. *Hint: use principal curvature coordinates.*

**Exercise 3.2.** The Gaussian curvature at a point $p$ can be defined as the limit

$$K_p = \lim_{\epsilon \to 0} \frac{A_G(B_\epsilon(p))}{A_M(B_\epsilon(p))}$$

where $B_\epsilon(p) \subset M$ is an $\epsilon > 0$ radius disk around $p$. $A_G$ measures the area of the Gauss map on $\mathbb{S}^2$ swept out by the normals in $B_\epsilon(p)$ while $A_M$ is the area of the surface associated with the $\epsilon$-ball around $p$. In other words, the Gaussian curvature is the limit of the quotient of Gauss area to surface area for shrinking areas surrounding $p$. Give an argument as to why this should be the case. You do not need to make detailed estimates of approximation, merely use arguments such as smoothness and continuity to argue that the integrals on the right approximate the value at a point on the left well.

Conclude that for triangle meshes *integrated* Gaussian curvature is associated with vertices and a good definition simply uses the area on the Gauss sphere spanned by the normals of the triangles adjacent to the given vertex.

### 3.0.1 Variational Origin of Curvature

In the curve case we saw definitions of curvature based on expressions for $dN$, the concept of osculating circle, and finally a variational derivation. In the surface case we will largely rely on variational definitions to derive simple expressions in the discrete case. So here are the smooth formulas.

**Theorem 3.1.** *Let $f : M \to \mathbb{R}^3$ be a parameterization of a smooth surface which bounds a volume $M = \partial\Omega$. Let $\sigma$ be the corresponding area 2-form,* i.e. *for each open set $U \subset M$ the area $A_U = A_{f(U)}$ of $f(U)$ is written as $A_U = \int_U \sigma$. Then the following variational identities hold*

$$\begin{aligned}
\dot{V}_M &= \int_M \langle \dot{f}, N\rangle \sigma \\
\dot{A}_M &= \int_M \langle \dot{f}, N\rangle 2H\sigma \\
\dot{H}_M &= \int_M \langle \dot{f}, N\rangle K\sigma \\
\dot{K}_M &= 0,
\end{aligned}$$

*where $V_M$, $A_M$, $H_M$, and $K_M$ denote the total volume enclosed, total area, total mean curvature, and total Gaussian curvature respectively and an overdot denotes the variation. From this we conclude that the gradient of volume gives us the area normal, the gradient of area twice the mean curvature normal, and the gradient of mean curvature the Gaussian curvature normal.*

Before we prove these identities we collect a number of equations that we'll use repeatedly.

Earlier we saw the definition of the wedge product of two vector valued 1-forms using the cross product as multiplication, *i.e.*, for $\alpha, \beta \in \Omega^1(M; \mathbb{R}^3)$ and tangent vectors $X, Y \in T_pM$

$$\alpha \wedge \beta(X,Y) := \alpha(X) \times \beta(Y) - \alpha(Y) \times \beta(X) = \beta \wedge \alpha(X,Y).$$

Thus we can write the area element on the embedded surface as

$$\sigma = \tfrac{1}{2}\langle N, df \wedge df\rangle$$

since

$$\begin{aligned}
\sigma(X,Y) &= \tfrac{1}{2}\langle N, df \wedge df(X,Y)\rangle \\
&= \tfrac{1}{2}\left(\det(N, df(X), df(Y)) - \det(N, df(Y), df(X))\right) \\
&= \det(N, df(X), df(Y)).
\end{aligned}$$

Using principal curvature coordinates allows us to simplify expressions involving $dN$ (since the result is independent of the coordinate system used, we might as well use the most convenient one). Let $X$ and $Y$ be tangent vectors along the principal curvature directions, *i.e.*, $dN(X) = \kappa_1\, df(X)$ and $dN(Y) = \kappa_2\, df(Y)$ then

$$dN \wedge df(X,Y) = H\, df \wedge df(X,Y)$$

$$dN \wedge dN(X,Y) = K\, df \wedge df(X,Y).$$

The volume as our first quantity takes a special place since it assumes that our surface has no boundary, $\partial M = \emptyset$, else it couldn't be the boundary of a region. For the other quantities, which start as integrals over the surface, we will not make the assumption of an empty boundary and thus have boundary terms appear as we apply Stokes' theorem in various places. Alright then, let's do it!

*Proof.* We begin with a formula for the volume written as a surface integral

$$V_M = \tfrac{1}{6}\int_M \langle f, df \wedge df\rangle.$$

To see this is true, recognize that $f(M)$ encloses a volume, which means that $f$ admits an extension defined on the interior $\Omega$. By writing $M = \partial\Omega$ and using Stokes' Theorem,

$$\tfrac{1}{6}\int_{\partial\Omega} \langle f, df \wedge df\rangle = \tfrac{1}{6}\int_{\Omega} \langle df \wedge, df \wedge df\rangle = V_M.$$

The last equality uses the fact that $\tfrac{1}{6}\langle df \wedge, df \wedge df\rangle$ is the $\mathbb{R}^3$ volume form, which can be seen by taking three vectors $X, Y, Z \in \mathbb{R}^3$:

$$\begin{aligned}
&\tfrac{1}{6}\langle df \wedge, df \wedge df\rangle(X,Y,Z)\\
&= \tfrac{1}{6}\langle df(X), df(Y) \times df(Z)\rangle - \tfrac{1}{6}\langle df(X), df(Z) \times df(Y)\rangle\\
&\quad + \tfrac{1}{6}\langle df(Y), df(Z) \times df(X)\rangle - \tfrac{1}{6}\langle df(Y), df(X) \times df(Z)\rangle\\
&\quad + \tfrac{1}{6}\langle df(Z), df(X) \times df(Y)\rangle - \tfrac{1}{6}\langle df(Z), df(Y) \times df(X)\rangle\\
&= \det\left(df(X), df(Y), df(Z)\right).
\end{aligned}$$

With this formula as a start, the variation of volume follows easily

$$\begin{aligned}
\dot{V}_M &= \tfrac{1}{6}\int_M \langle \dot{f}, df \wedge df\rangle + \tfrac{1}{3}\int_M \langle f, df \wedge d\dot{f}\rangle\\
&= \tfrac{1}{2}\int_M \langle \dot{f}, df \wedge df\rangle = \int_M \langle \dot{f}, N\rangle \sigma.
\end{aligned}$$

The first equality is due to the product rule and $df \wedge d\dot{f} = d\dot{f} \wedge df$. The second equality follows from integration by parts

$$d\det(f, df, \dot{f}) = \langle df \wedge df, \dot{f}\rangle - \langle f, df \wedge d\dot{f}\rangle$$

and subsequent application of Stokes' theorem (with $\partial M = \emptyset$). From now on we will allow $\partial M \neq \emptyset$.

On to area

$$A_M = \tfrac{1}{2}\int_M \langle N, df \wedge df\rangle.$$

The variation follows as

$$\begin{aligned}
\dot{A}_M &= \tfrac{1}{2}\int_M \langle \dot{N}, df \wedge df\rangle + \int_M \langle N, df \wedge d\dot{f}\rangle \\
&= \int_M \langle \dot{f}, dN \wedge df\rangle - \int_M d\det(N, df, \dot{f}) \\
&= \int_M \langle \dot{f}, df \wedge df\rangle H - \int_{\partial M} \langle \dot{f}, N \times df\rangle \\
&= \int_M \langle \dot{f}, N\rangle 2H\sigma - \int_{\partial M} \langle \dot{f}, N \times df\rangle.
\end{aligned}$$

The first equality results again from the product rule. Here the first term vanishes because $\dot{N}$ lies in the tangent plane (is linearly dependent with $df$), while the second term is transformed through integration by parts

$$d\det(N, df, \dot{f}) = \langle dN \wedge df, \dot{f}\rangle - \langle N, df \wedge d\dot{f}\rangle.$$

The third equality uses $dN \wedge df = H\, df \wedge df$ (see above), and Stokes' theorem. Not surprisingly the boundary term corresponds to the intuition that variation of the boundary curve in the outward direction increases area the most.

For the total mean curvature $H_M$ we get

$$H_M = \int_M H\sigma = \tfrac{1}{2}\int_M \langle N, df \wedge dN\rangle$$

by virtue of our earlier observation that $df \wedge dN = H\, df \wedge df$. Consequently the variation is given by

$$\begin{aligned}
\dot{H}_M &= \tfrac{1}{2}\int_M \langle N, d\dot{f} \wedge dN\rangle + \tfrac{1}{2}\int_M \langle N, df \wedge d\dot{N}\rangle \\
&= \tfrac{1}{2}\int_M \langle \dot{f}, dN \wedge dN\rangle - \tfrac{1}{2}\int_{\partial M} \big(\det(N, dN, \dot{f}) + \det(N, df, \dot{N})\big)
\end{aligned}$$

$$= \int_M \langle \dot{f}, N \rangle K\sigma - \tfrac{1}{2} \int_{\partial M} \big( \langle \dot{f}, N \times dN \rangle + \langle \dot{N}, N \times df \rangle \big).$$

The first equality follows from the product rule and the fact that $\langle \dot{N}, df \wedge dN \rangle = 0$ since $\dot{N}$, $df$, as well as $dN$ all lie in the 2D tangent space. The second equality exploits integration by parts

$$d\det(N, dN, \dot{f}) = \langle dN \wedge dN, \dot{f} \rangle - \langle N, dN \wedge d\dot{f} \rangle$$
$$d\det(N, df, \dot{N}) = -\langle N, df \wedge d\dot{N} \rangle.$$

The third equality takes advantage of $dN \wedge dN = K\, df \wedge df$ (see above). (Who has a good geometric interpretation for these boundary terms?)

Finally, for the total Gauss curvature $K_M$ we get

$$K_M = \int_M K\sigma = \tfrac{1}{2} \int_M \langle N, dN \wedge dN \rangle$$

since $dN \wedge dN = K\, df \wedge df$. For the variation we find

$$\begin{aligned} \dot{K}_M &= \int_M \langle N, dN \wedge d\dot{N} \rangle \\ &= -\int_{\partial M} \det(N, dN, \dot{N}) \\ &= -\int_{\partial M} \langle \dot{N}, N \times dN \rangle. \end{aligned}$$

The first equality follows from the product rule followed by the observation that $\dot{N}$ is orthogonal to $dN \wedge dN$ and the fact that $dN \wedge d\dot{N} = d\dot{N} \wedge dN$. The second equality follows from integration by parts

$$d\det(N, dN, \dot{N}) = -\langle N, dN \wedge d\dot{N} \rangle.$$

In particular this result shows that total Gauss curvature is constant for surfaces without boundary, a version of the Gauss-Bonnet theorem. □

At this point we have worked out smooth formulas involving area, mean curvature, and Gauss curvature normals as variations of volume, area, and mean curvature respectively. To make this machinery work for the discrete setting we have two choices now. We can take the smooth formulas and evaluate them over small regions (effectively forming averages) to define the corresponding normals. This step can be further enhanced by using Stokes' theorem to turn area integrals into boundary integrals (around the averaging region). Incidentally, the Stokes transformation removes one order of differentiation from the integrand, making life easier in the discrete setting where we apply these formulas to a piecewise linear mesh. Another path is to first discretize and then vary. In that setting we look at the variation of volume, area, and mean curvature with respect to moving a vertex. The result is almost the same and the subject of the following two exercises.

**Exercise 3.3** (Conservation laws of curvature normals)**.** When a surface is not sufficiently smooth, *e.g.*, pointwise evaluation of $N$ is not available, it is often instructive to look at the integral of the quantity of interest over a neighborhood $U \subset M$. Using Stokes' Theorem, show that

$$\int_U N\sigma = \tfrac{1}{2}\int_{\partial U} f \times df$$
$$\int_U HN\sigma = \tfrac{1}{2}\int_{\partial U} N \times df$$
$$\int_U KN\sigma = \tfrac{1}{2}\int_{\partial U} N \times dN,$$

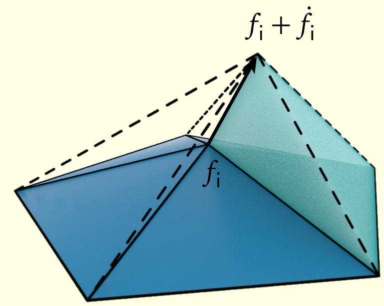
*i.e.*, they can be written as boundary integrals.

**Exercise 3.4** (Discrete volume gradient)**.** Let $f : \{p_0, \ldots, p_{n-1}\} \to \mathbb{R}^3$ be the point (vertex) positions of a closed triangulated surface. Let $V$ be the total polytope volume enclosed by the discrete surface. Let $A_{\mathsf{ijk}}$ and $N_{\mathsf{ijk}}$ be the area and the outer normal of the triangle $t_{\mathsf{ijk}}$. Show that under a variation $\dot{f} : \{p_0, \ldots, p_{n-1}\} \to \mathbb{R}^3$, the variation of the total volume is

$$\dot{V} = \sum_{p_{\mathsf{i}}} \sum_{t_{\mathsf{ijk}} \succ p_{\mathsf{i}}} \left(\tfrac{1}{3} A_{\mathsf{ijk}} N_{\mathsf{ijk}}\right) \dot{f}_{\mathsf{i}},$$

where $t_{\mathsf{ijk}} \succ p_{\mathsf{i}}$ denotes $p_{\mathsf{i}}$ is incident to $t_{\mathsf{ijk}}$. That is, the point normal derived from the volume gradient is the area-weighted averaging of the neighboring face normals.

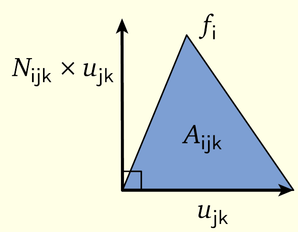
*Hint: What is the extra volume gained by moving one vertex?*




**Exercise 3.5** (Discrete area gradient)**.** Let $f : \{p_0, \ldots, p_{n-1}\} \to \mathbb{R}^3$ be the point (vertex) positions of a triangulated surface. Let $A$ be the total area of the discrete surface. Denote $A_{ijk}$, $N_{ijk}$ the area and normal of the triangle $t_{ijk}$.

(a) For $p_i \prec t_{ijk}$, let $e_{jk}$ be the edge in $t_{ijk}$ across from $p_i$. Let $u_{jk} = f_k - f_j \in \mathbb{R}^3$ be the edge vector of $e_{jk}$. Show that the gradient of the area with respect to $f_i$ is

$$\frac{\partial A_{ijk}}{\partial f_i} = \tfrac{1}{2} N_{ijk} \times u_{jk}.$$




*Hint: Which direction should you "push" $f_i$ in order to increase $A_{ijk}$ most efficiently?*

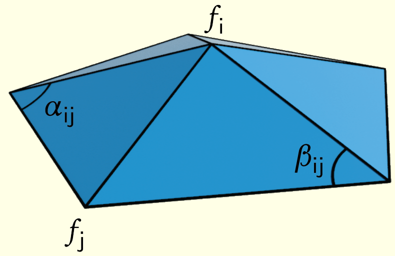
(b) The variation of total area with respect to $f_i$ is

$$\frac{\partial A}{\partial f_i} = \sum_{t_{ijk} \succ p_i} \frac{\partial A_{ijk}}{\partial f_i}.$$

Show that it can be expressed as

$$\frac{\partial A}{\partial f_i} = \tfrac{1}{2} \sum_{e_{ij} \prec p_i} \left(\cot \alpha_{ij} + \cot \beta_{ij}\right)\left(f_i - f_j\right).$$

where $\alpha_{ij}$ and $\beta_{ij}$ are the angles across from $e_{ij}$ in the two triangles incident to $e_{ij}$.




*Hint: Express the vectors of interest using a coordinate basis containing the edge* $(f_i - f_j)$.

(c) Directly from (a), observe that

$$\frac{\partial A}{\partial f_i} = \frac{1}{2} \sum_{t_{ijk} \succ p_i} N_{ijk} \times u_{jk}.$$

Using the result $\int_U HN\sigma = \frac{1}{2}\int_{\partial U} N \times df$ from Exercise 3.3, argue that $\frac{\partial A}{\partial f_i}$ is the integral of mean curvature normal $HN$ over the one-ring neighborhood of $p_i$.

**Exercise 3.6** (Visualization of mean and Gaussian curvature)**.** In this exercise you will visualize the mean curvature and the Gaussian curvature of a given discrete surface.

(a) From Exercise 3.5, the discrete mean curvature normal on a point $p_i$ is given by

$$(HN)_i = \frac{1}{2} \sum_{t_{ijk} \succ p_i} N_{ijk} \times u_{jk}$$

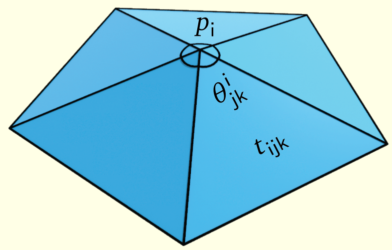
where $u_{jk}$ is the edge vector of the edge in $t_{ijk}$ across from $p_i$. It is understood as that $(HN)_i = \int_{U_i} HN\sigma$ where $U_i$ is the one-ring neighborhood of $p_i$. While the direction of $(HN)_i$ tells us a notion of discrete normal, the magnitude of $(HN)_i$ tells us the mean curvature $H$ *only up to a sign*. Finding the sign amounts to finding a consistent orientation for $N$ across neighboring points. To keep it simple, you can take $N_i^V$ the area weighted normals (volume gradient) and determine the sign using $\operatorname{sgn}\langle HN_i, N_i^V\rangle$. Visualize (with **Houdini**) the mean curvature by assigning colors to points according to

$$H_i = \pm \left| \frac{\int_{U_i} HN\sigma}{\int_{U_i} \sigma} \right|.$$

(b) The discrete Gaussian curvature given on each point $p_{\mathrm{i}}$ is given by

$$K_{\mathrm{i}} = 2\pi - \sum_{t_{\mathrm{ijk}} \succ p_{\mathrm{i}}} \theta^{\mathrm{i}}_{\mathrm{jk}}$$

where $\theta^{\mathrm{i}}_{\mathrm{jk}}$ is the angle of the triangle $t_{\mathrm{ijk}}$ at the vertex attached to $p_{\mathrm{i}}$.




In fact the angle defect $K_{\mathrm{i}}$ is the area of the spherical polygon enclosed by great arcs connecting the vertices $N_{\mathrm{ijk}}$ for $t_{\mathrm{ijk}} \succ p_{\mathrm{i}}$, *i.e.* $K_{\mathrm{i}} = \int_{U_{\mathrm{i}}} K\sigma$ over the one-ring neighborhood $U_{\mathrm{i}}$ of $p_{\mathrm{i}}$. Visualize (with **Houdini**) the Gaussian curvature by assigning colors to points according to the value of $\frac{K_{\mathrm{i}}}{\int_{U_{\mathrm{i}}} \sigma}$.