Chapter 5

Differential Equations on Manifolds

In this chapter we look at a few classical linear differential equations formulated using the exterior calculus. They include heat equation, Poisson equation, Maxwell's equations.

These differential equations originate from classical physics and are often formulated in coordinates. For example, the Laplace operator Δ, which you will see appearing in a lot of the equations we consider, applied to a function $u\colon \mathbb{R}^3 \to \mathbb{R}$ is expressed with $\Delta u = \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u + \frac{\partial^2}{\partial z^2}u$. However, when the coordinate $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$ one uses is not orthonormal everywhere, which is often the case on a manifold, fundamental operators such as the Laplacian expressed in coordinates becomes impenetrable:

$$\Delta u = \frac{1}{\sqrt{\det g}} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial}{\partial x^i}\left(\sqrt{\det g}(g^{-1})_{ij}\frac{\partial}{\partial x^j}u\right), \quad g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle,$$

to say nothing of its discretization for computation. Compare it with the coordinate-free exterior calculus expression

$$\Delta u = -\delta du,$$

which is often physically, geometrically, and numerically (DEC) more insightful.

Physical Units

Many of the equations come from physics, and we will write the precise physical unit in these cases. A differential k-form is regarded as an *oriented k-density*, namely it describes some physical quantity per unit k-dimensional volume. For example, in $\mathbb{R}^3$, the mass density ρ is a 3-form which becomes a mass after integration over a volume. We denote

$$\rho \in \Omega^3(M; \mathbb{R}\,\mathrm{kg})$$

and read that ρ is a *mass*-valued (or kg-valued) 3-form. The actual physical dimension of ρ as a mass density is

$$[\rho] = [{}^{\mathrm{kg}}/_{\mathrm{m}^3}]$$

since it has to integrate over a 3-dimensional region to become an element of $\mathbb{R}\,\mathrm{kg}$. In other words, one has to take 3 tangent vectors X_1, X_2, X_3 to plug in into the alternating triple linear form ρ to evaluate the mass, and tangent vectors have a unit of meter.

Another example is that a mass flux $J \in \Omega^2(M; \mathbb{R}\,{}^{\mathrm{kg}}/_{\mathrm{s}})$ is a 2-form modeling the amount of mass passing through a surface per unit area per unit time. The unit of J is $[J] = [{}^{\mathrm{kg}}/_{\mathrm{m}^2\mathrm{s}}]$, and its evaluation after integrating over an area is a number in $\mathbb{R}\,{}^{\mathrm{kg}}/_{\mathrm{s}}$.

In general, a differential k-form $\omega \in \Omega^k(M; \mathbb{R}\,\mathrm{unit})$ has a unit $[\omega] = [{}^{\mathrm{unit}}/_{\mathrm{m}^k}]$.

The exterior derivative d maps $\Omega^k(M; \mathbb{R}\,\mathrm{unit})$ to $\Omega^{k+1}(M; \mathbb{R}\,\mathrm{unit})$. In other words, the exterior derivative gives an extra "length" in the denominator, $[d\omega] = [\omega][{}^{1}/_{\mathrm{m}}]$. For this being said, in $\mathbb{R}^n$ the coordinate functions $x^1, \dots, x^n$, which are meter-valued functions $: M \to \mathbb{R}\,\mathrm{m}$, gives a basis for k-form $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ that belongs to $\Omega^k(M; \mathbb{R}\,\mathrm{m}^k)$. Therefore $[dx^I] = [1]$ has no unit. The unit of $\omega = \sum_I w_I\, dx^I$ is the same as that of the coefficient functions w_I.

The standard Hodge star $*$ has no unit, since $*\omega = \sum_I w_I(*dx^I)$ has the same unit as w_I and thus the same as the unit of ω. Therefore

$$*_k\colon \Omega^k(M; \mathbb{R}\,\mathrm{unit}) \to \Omega^{n-k}(M; \mathbb{R}\,\mathrm{unit}\cdot \mathrm{m}^{n-2k})$$

in order to give $[\omega] = [*\omega] = [{}^{\mathrm{unit}}/_{\mathrm{m}^k}]$.

In many physical examples, we will take the Hodge star as a more general operator that can map differential forms of different unit of integrals:

$$*_k\colon \Omega^k(M; \mathbb{R}\,\mathrm{unit}_1) \to \Omega^{n-k}(M; \mathbb{R}\,\mathrm{unit}_2)$$

encoding a material property that relates ${}^{\mathrm{unit}_1}/_{\mathrm{m}^k}$ with ${}^{\mathrm{unit}_2}/_{\mathrm{m}^{n-k}}$.

5.1 Heat Equation

The heat equation, a.k.a. diffusion equation, models how temperature diffuses in a material. Consider the domain to be an n-dimensional manifold M, and at each $t \geq 0$ let $u(t)\colon M \to \mathbb{R}^+\,\mathrm{K}$ be the temperature (in Kelvin) at each point on M. The 1-form $du \in \Omega^1(M; \mathbb{R}\,\mathrm{K})$ describes the temperature difference: $\int_\gamma du$ is the temperature difference across a path γ. **Fourier's Law of thermal conductance** (or **Newton's Law of cooling**) states that the time rate of heat transfer is proportional to the (negative) temperature difference. Namely, the **heat flux**, the energy flowing through a surface per unit area per unit time, is a $(n-1)$-form $Q(t) \in \Omega^{n-1}(M; \mathbb{R}\,\mathrm{J/s})$ that is related to du through

$$Q = -*du. \tag{5.1}$$

Here the Hodge dual $* = *_1$ maps a *temperature*-valued 1-form to an *energy-per-second*-valued $(n-1)$-form. In particular $*_1$ serves the purpose of the **thermal conductivity**.

To illustrate this, we consider the case of isotropic and uniform thermal conductivity in the material. In this case, we look at a primal edge e and its corresponding dual facet e^* in a discrete setup, and discover

$$\int_{e^*} Q = -\int_{e^*} *du = -\frac{|e^*|}{|e|}\int_e du.$$

On the left hand side we have the total heat transfer per unit time, which is negative proportional to the temperature difference on the right hand side. The conductance rate is $\frac{|e^*|}{|e|}$, which is proportional to the cross section area $|e^*|$ and inverse proportional to the distance $|e|$. This is the familiar empirical thermal conductivity described in terms of the material geometry.

The above example uses $* = *_1$ derived from a standard metric on M. In general one can take $* = *_1$ in $Q = -*du$ at each point p an arbitrary linear map

$$*_1 \colon \Omega^1_p(M; \mathbb{R}\,\mathrm{K}) \to \Omega^{n-1}_p(M; \mathbb{R}\,\mathrm{J/s})$$

with positive orientation, *i.e.* $(\alpha \wedge *_1\alpha)$ is a positive n-form. It encodes a general thermal conductivity in a material.

With a heat flux Q given, the change in temperature over time is given by

$$*\tfrac{\partial}{\partial t}u = -dQ. \tag{5.2}$$

Here $* = *_0$ maps changes in temperature to heat flux. That is,

$$*_0 \colon \Omega^0(M; \mathbb{R}\,\mathrm{K/s}) \to \Omega^n(M; \mathbb{R}\,\mathrm{J/s})$$

is the **heat capacity**. In a material with uniform specific heat, on a small region U

$$-\int_{\partial U} Q = \int_U *\tfrac{\partial}{\partial t}u \approx |U|\tfrac{\partial}{\partial t}u$$

which states that the total in-flowing energy (LHS) results in a rise in temperature (RHS) with a rate proportional to the volume $|U|$. In general, at each $p \in M$, $*_0$ can be an arbitrary linear map $*_0 \colon \Omega^0(M; \mathbb{R}\,\mathrm{K/s}) \to \Omega^n(M; \mathbb{R}\,\mathrm{J/s})$ with positive $*_0 1$, modeling a general local heat capacity in a material.

Putting (5.1) (Newton-Fourier's Law) and (5.2) (conservation of energy) together, one obtains the **heat equation**

$$\tfrac{\partial}{\partial t}u = *_0^{-1}\, d *_1 du = \Delta u. \tag{5.3}$$

Exercise 5.1 (Conservation of energy)**.** Here is a quick exercise for using Stokes' Theorem. Suppose M has no boundary. Show that the total internal (thermal) energy $\int_M *_0 u$ is constant in time when u satisfies (5.3).

If there is a heat source, namely there is $\beta(t) \in \Omega^n(M; \mathbb{R}^{\mathrm{J}/\mathrm{s}})$ being the rate of heat transferred into the domain per unit volume and per unit time, then (5.2) is replaced with

$$*\tfrac{\partial}{\partial t} u = -dQ + \beta$$

and (5.3) becomes

$$\tfrac{\partial}{\partial t} u = \Delta u + *_0^{-1} \beta. \tag{5.4}$$

How should the temperature u behave subject to the heat equation? Intuitively, the heat keeps flowing from high temperature to low temperature, which effectively keeps averaging the temperature. This process is also called a **diffusion process**. As the averaging process suggests, we expect the following properties for the heat equation, which will be shown as the discussion goes on.

- **Maximum principle**. For u satisfying the heat equation for $t \geq 0$, the maximum and minimum temperature occurs at $t = 0$ or on the boundary ∂M.
- **Time irreversibility**. The total entropy increases in time.
- **Dirichlet energy** $\int_M *|du|^2$, which measures the roughness of the function u, decreases under heat flow.

5.1.1 Maximum Principle

The maximum principle for the heat equation states that the maximum of the temperature occurs initially or on the boundary, as expected in an averaging process. With the maximum/minimum principle, we justify that the temperature in Kelvin $u \geq 0$ is not violated if $u \geq 0$ initially and on the boundary.

Theorem 5.1 (Maximum principle)**.** *Suppose u satisfies* (5.3) *for* $0 < t \leq T$ *on a compact manifold M. Then the maximum of u occurs on $t = 0$ or on ∂M.*

Proof. First of all, by compactness of $M \times [0, T]$ the maximum is always attained. Let us first prove a slight variation of the statement: if $\tilde{u}$ satisfies

$$\tfrac{\partial}{\partial t} \tilde{u} < \Delta \tilde{u}, \quad \text{for } 0 < t \leq T,$$

then its maximum must occurs on $t = 0$ or on ∂M. To show this, assume that there is an interior point $p_0 \in M$ and $0 < t_0 \leq T$ so that $\tilde{u}_{(p_0, t_0)}$ is a maximum. Then $\tfrac{\partial \tilde{u}}{\partial t}_{(p_0, t_0)} \geq 0$,

$d\tilde{u}_{(p_0,t_0)} = 0$ and $\Delta u_{(p_0,t_0)} \le 0$ (Can you see why? If they do not hold, take a small neighborhood and integrate over it to show that u reaches a larger value somewhere else). We obtain a contradiction $0 \le \frac{\partial \tilde{u}}{\partial t}_{(p_0,t_0)} < \Delta \tilde{u}_{(p_0,t_0)} \le 0$.

Now to show the maximum principle for u solving $\frac{\partial u}{\partial t} = \Delta u$ for $0 < t \le T$, take $\tilde{u}_\varepsilon := u - \varepsilon t$ for each $\varepsilon > 0$. Note that $\frac{\partial \tilde{u}_\varepsilon}{\partial t} < \Delta \tilde{u}_\varepsilon$ hence its maximum occurs only on $t = 0$ or ∂M. Take $\varepsilon \to 0$, we obtain $\tilde{u}_\varepsilon \to u$ and

$$\max_{M \times [0,T]} u = \max_{\partial M \cup \{t=0\}} u.$$

□

Corollary 5.1. *By a similar proof, the minimum occurs also on $t = 0$ or on ∂M.*

Exercise 5.2 (Uniqueness)**.** On a compact manifold M without boundary, using the maximum (and minimum) principle, show that the solution to (5.3) is uniquely determined by its initial condition $u_{t=0}$.

Hint: Consider u, v both satisfying the heat equation with common initial condition. What happens to their difference?

5.1.2 Entropy

An important concept in the thermodynamics is the **entropy**. An entropy is a quantity of a system that only increases (or decreases depending on convention) in time, indicating **time-irreversibility** of the system. A heat diffusion process governed by (5.3) does have (a lot of different definitions of, see Exercise 5.3,) entropy that breaks the time-reversibility symmetry.

The standard entropy introduced in the classical thermodynamics is defined through the following. The entropy density is an n-form $\sigma \in \Omega^n(M; \mathbb{R}\,\mathrm{J}/\mathrm{K})$ whose changes in time is given by

$$\frac{\partial}{\partial t}\sigma := -\frac{1}{u}\, dQ.$$

Recall that $Q \in \Omega^{n-1}(M; \mathbb{R}\,\mathrm{J}/\mathrm{s})$ is the heat flux (energy per area per time), and $-dQ$ is the total in-flowing energy rate. From (5.2) we have

$$\frac{\partial}{\partial t}\sigma = *\frac{1}{u}\frac{\partial}{\partial t}u = *\frac{\partial}{\partial t}(\log u),$$

which suggests that

$$\sigma = *\log u$$

up to an additive constant. The **total entropy** is given by $S = \int_M \sigma$. Assuming that M has no boundary and there is no heat source, the change of total entropy is non-negative (**Second Law of thermodynamics**):

$$\begin{aligned}\tfrac{d}{dt}S &= \int_M \tfrac{\partial}{\partial t}\sigma = \int_M -\tfrac{1}{u}\,dQ \overset{(5.1)}{=} \int_M \tfrac{1}{u}\,d{*}du \\ &= -\int_M d\left(\tfrac{1}{u}\right)\wedge {*}du = \int_M \frac{1}{u^2}\,du\wedge{*}du = \int_M {*}\tfrac{1}{u^2}|du|^2 \geq 0\end{aligned}$$

and $\frac{d}{dt}S = 0$ if and only if $du \equiv 0$, *i.e.* u is a constant (at thermal equilibrium). Here we have used integration by parts derived from $d\left(\frac{1}{u}{*}du\right) = d\left(\frac{1}{u}\right)\wedge{*}du + \frac{1}{u}\,d{*}du$ and Stokes' Theorem that removes the boundary term.

Exercise 5.3 (Second Law of Thermodynamics)**.** Let $f : \mathbb{R}^+ \to \mathbb{R}$ be any smooth strictly convex function, *i.e.* $f'' > 0$. Such functions are also known as *mathematicians' entropy function*, whereas the special case $f = -\log$ is *minus the physical entropy function*. Show that the total entropy

$$S(u) := \int_M {*}f(u)$$

is strictly *decreasing* in time when u is non-constant and satisfies (5.3).

Hint: Integration by parts.

Exercise 5.4 (L^2 function)**.** Show that if u satisfies (5.3) on M without boundary, and if $u \in L^2(M) = \left\{v : M \to \mathbb{R} \,\middle|\, \int_M {*}v^2 < \infty\right\}$ at $t = 0$, then $u \in L^2(M)$ for all later time.

Hint: Exercise 5.3.

5.1.3 Dirichlet Energy

Here is a whole different way of looking at the heat equation. For each function $u\colon M \to \mathbb{R}$, the **Dirichlet energy**

$$E_D(u) = \tfrac{1}{2}\int_M du\wedge{*}du = \tfrac{1}{2}\|du\|^2_{L^2}$$

is a canonical way to measure the "roughness" of the function u. When $E_D(u) = 0$, we have u being constant (in each connected component of M). In a diffusion process, *e.g.* governed by the heat flow (5.3), we expect the function u becomes smoother in time. This is indeed true measured with the Dirichlet energy: assuming $\partial M = \emptyset$,

$$\tfrac{d}{dt}E_D(u) = \int_M d\left(\tfrac{\partial u}{\partial t}\right)\wedge * du = \int_M \tfrac{\partial u}{\partial t} *(-\Delta u) = -\int_M *\left(\tfrac{\partial u}{\partial t}\right)^2 \leq 0$$

and E_D is constant if and only if u is stationary.

A more interesting fact between Dirichlet energy and the heat equation is that the heat flow is the **steepest descent** (**gradient flow**) of the Dirichlet energy. Given a function $u\colon M \to \mathbb{R}$ and consider a variation $\dot{u}$ with $\dot{u}|_{\partial M} = 0$. Then the variation of its Dirichlet energy is

$$\dot{E}_D = \int_M d\dot{u} \wedge * du = \int_M - * \dot{u}\Delta u.$$

That is, Δu is the negative gradient of the Dirichlet energy, and (5.3) is understood as the gradient flow of the Dirichlet energy.

One can similarly show that the heat equation with heat source (5.4) is the gradient flow of

$$E_D(u) + \int_M u\beta.$$

5.1.4 Weak formulation

For u to satisfy $\frac{\partial}{\partial t}u = \Delta u$, seemingly u requires twice differentiability (due to the Laplacian). Nonetheless, there is an alternative way to "describe" the heat equation with at most one differentiation. Multiply the heat equation by any differentiable function $\varphi\colon M \to \mathbb{R}$ and integrate the result over M:

$$\int_M *\varphi \tfrac{\partial}{\partial t} u = \int_M *\varphi \Delta u = -\int_M d\varphi \wedge * du.$$

We again used the integration by parts, assuming M has no boundary. If φ are time-independent, we may pull out $\frac{\partial}{\partial t}$ from the integral and obtain

$$\tfrac{d}{dt}\langle\!\langle \varphi, u\rangle\!\rangle = -\langle\!\langle d\varphi, du\rangle\!\rangle, \quad \varphi \in \Omega^0(M;\mathbb{R}). \tag{5.5}$$

We say u is a **weak solution** to the heat equation if u satisfies the **heat equation in the weak form** (5.5) for all **test functions** $\varphi \in \Omega^0(M;\mathbb{R})$. If u satisfies (5.3), *i.e.* u is a **strong solution**, then u must also be a weak solution (as just derived). If u is a weak solution, and if one could show that u is twice differentiable (**regularity**), then u is also a strong solution (by reversing the integration by parts).

The weak formulation is important not only in PDE analysis but also in numerical approaches (as the base of finite element method). With the knowledge that the strong solution exists and uniquely agrees with the weak solution, we are safe to adopt numerical approaches which solves the weak form, say, using piecewise linear test functions which are only once differentiable.

5.1.5 Discrete heat equation

Using DEC it is rather straightforward to write down a heat equation on a mesh. In fact, since the entire heat equation theory involves only primal 0-forms (temperature u), 1-forms (du), dual $(n-1)$-forms (energy flux) and dual n-forms (energy density), the heat equation requires only points and edges, such as a graph. But here we consider a discrete complex M such as a triangulated surface.

Let the temperature u sit on points (vertices). Hence $u \in C^0(M;\mathbb{R}\,\mathrm{K})$, where C^k denotes the k-cochain. The temperature difference $d_0 u \in C^1(M;\mathbb{R}\,\mathrm{K})$ sits on edges, where $d_0 = \partial_1^{\intercal}$ is the incidence matrix. Now, consider the diagonal Hodge star $*_1$ be a diagonal matrix with positive entries. These entries are the **thermal conductivity**, or the **edge weight** in general. Namely,

$$Q = -*_1 d_0 u \in C^{n-1}(M^*;\mathbb{R}\,\mathrm{J/s})$$

is the heat flux sitting on the dual $(n-1)$-cells across the primal edges. When considering a homogeneous material with uniform isotropic heat conductivity, we take the diagonal Hodge star

$$*_1 = \operatorname{diag}\left(\frac{|e^*|}{|e|}\right),$$

i.e., the cotangent weight in case of a triangulated surface. The total in-flowing flux into a dual n-cell is given by

$$-d_0^{\intercal} *_1 d_0 u.$$

Finally, take the primal Hodge star on the 0-cell $*_0$, which is the mass matrix. In the context of heat equation, $*_0$ is the **heat capacitance**. With this final piece we arrive at the discrete heat equation

$$*_0 \tfrac{\partial}{\partial t} u = -d_0^{\intercal} *_1 d_0 u. \tag{5.6}$$

Similar to the smooth setting, we may also consider a heat source $\beta \in C^n(M^*;\mathbb{R}\,\mathrm{J/s})$ sitting on the dual n-cell as the amount of additional heat poured into the cell per unit time:

$$*_0 \tfrac{\partial}{\partial t} u = -d_0^{\intercal} *_1 d_0 u + \beta. \tag{5.7}$$

Analogous to the smooth theory, the discrete heat equation is the gradient flow to the Dirichlet energy. Here one has to be careful that on $C^0(M)$, which is the vector space u belongs to, uses an inner product respecting $*_0$. That is, for $f, g \in C^0(M)$, we have $\langle\!\langle f, g\rangle\!\rangle_{C^0(M)} = f^{\intercal} *_0 g$. Similarly, for $\xi, \eta \in C^1(M)$, $\langle\!\langle \xi, \eta\rangle\!\rangle_{C^1(M)} = \xi^{\intercal} *_1 \eta$. Now, the Dirichlet energy is the L^2-norm of $d_0 u$, which is given by

$$E_D = \tfrac{1}{2}\langle\!\langle d_0 u, d_0 u\rangle\!\rangle_{C^1(M)}$$

$$= \tfrac{1}{2} u^{\intercal} d_0^{\intercal} *_1 d_0 u.$$

Its variation is then

$$\dot{E}_D = \dot{u}^{\intercal} d_0^{\intercal} *_1 d_0 u = \langle\!\langle \dot{u}, *_0^{-1} d_0^{\intercal} *_1 d_0 u \rangle\!\rangle_{C^0(M)}.$$

Hence

$$\operatorname{grad}_{C^0(M)} E_D = *_0^{-1} d_0^{\intercal} *_1 d_0 u$$

which justifies that (5.6) is the gradient flow of the discrete Dirichlet energy.

Finite Element Approach

The discrete heat equation (5.6) and (5.7) coincides with the discretization obtained from the piecewise linear (PL) finite element method (FEM) on a triangulated surface. The PL-FEM approximates the solution using a piecewise linear function. To derive a finite element scheme, we begin with the heat equation in the weak form (5.5). To be precise, the weak heat equation (5.5) is a statement for u, φ in the function space

$$W^{1,2}\Omega^0(M;\mathbb{R}) = \left\{ v \in \Omega^0(M;\mathbb{R}) \,\middle|\, \|v\|^2 < \infty \text{ and } \|dv\|^2 < \infty \right\}.$$

(The notation $W^{k,p}$ indicates k-times differentiability and $\int |\cdot|^p < \infty$ integrability.) Observe that the space $\mathrm{PL}\Omega^0(M;\mathbb{R})$ of all functions which are continuous and piecewise linear over the faces in a triangular mesh is a *finite dimensional* subspace of this infinite dimensional space $W^{1,2}\Omega^0(M;\mathbb{R})$. A canonical basis for $\mathrm{PL}\Omega^0(M;\mathbb{R})$ is the "hat functions" $\{\phi_{\mathrm{i}}\}_{\mathrm{i}}$ which are piecewise linear and $\phi_{\mathrm{i}}(p_{\mathrm{j}}) = \delta_{\mathrm{ij}}$ on vertex p_{j} (Figure 5.1).

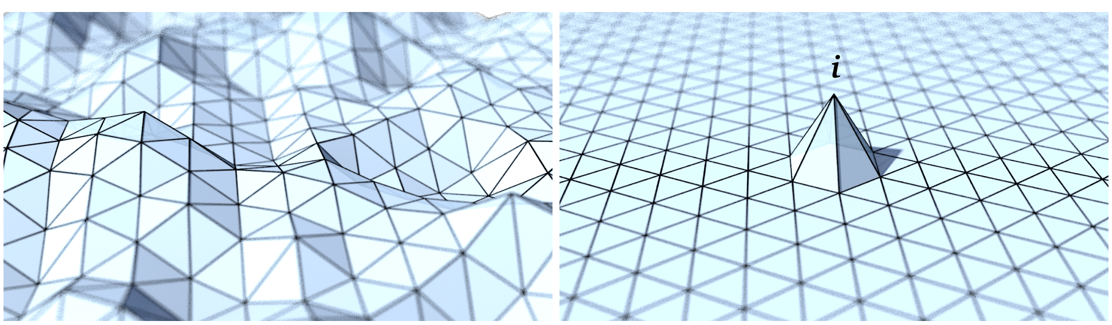


Figure 5.1: *Left: a generic piecewise linear function whose linear pieces are the triangle faces of a given mesh. Right: the hat function ϕ_i.*

Any function $\hat{u} \in \mathrm{PL}\Omega^0(M;\mathbb{R})$ is written as

$$\hat{u} = \sum_{\mathrm{i}} u_{\mathrm{i}} \phi_{\mathrm{i}}$$

where each $u_i \in \mathbb{R}$ is a number sitting on a vertex, *i.e.* $u = (u_i)_i \in C^0(M;\mathbb{R})$ zeroth cochain. Notice that $\hat{u}(p_i) = u_i$. Now, to best approximate the heat equation, we *orthogonally project* the heat equation onto the subspace $\mathrm{PL}\Omega^0(M;\mathbb{R})$:

$$\tfrac{d}{dt}\langle\langle \phi_i, \hat{u}\rangle\rangle = -\langle\langle d\phi_i, d\hat{u}\rangle\rangle, \quad \forall \text{ basis element } \phi_i.$$

Note that this is the orthogonal projection of the heat equation in the sense that the remainder error $r =$"$\frac{\partial}{\partial t}\hat{u} - \Delta\hat{u}$" is L^2-orthogonal to all basis element ϕ_i.

Now, with $\hat{u} = \sum_j u_j \phi_j$, we have

$$\sum_j \langle\langle \phi_i, \phi_j\rangle\rangle \tfrac{d}{dt} u_j = \sum_j -\langle\langle d\phi_i, d\phi_j\rangle\rangle u_j, \quad \text{for all i.} \tag{5.8}$$

By working out the L^2 inner product (in the smooth sense) for $\langle\langle \phi_i, \phi_j\rangle\rangle$ and $\langle\langle d\phi_i, d\phi_j\rangle\rangle$, define matrices $(*_0^{\mathrm{FEM}})_{ij} = \langle\langle \phi_i, \phi_j\rangle\rangle$ (the **mass matrix**) and $L_{ij} = -\langle\langle d\phi_i, d\phi_j\rangle\rangle$ (the **stiffness matrix**). Thus (5.8) becomes

$$*_0^{\mathrm{FEM}} \tfrac{d}{dt} u = Lu. \tag{5.9}$$

If we can compute the matrix elements of L and $*_0^{\mathrm{FEM}}$, we can solve (5.9) for the coefficients u.

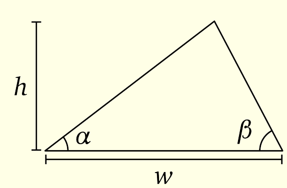
Exercise 5.5 (The stiffness matrix)**.** On a discrete triangular mesh M, the stiffness matrix L is given by

$$L_{ij} = -\int_M d\varphi_i \wedge * d\varphi_j = -\sum_{t_{ijk} \succ e_{ij}} \langle \operatorname{grad}\varphi_i, \operatorname{grad}\varphi_j\rangle A_{ijk}$$

where A_{ijk} is the area of triangle t_{ijk}. Here we used the fact that the hat functions φ_i, φ_j are piecewise linear and hence their gradients are piecewise constant on the triangles. We also used the fact that $\operatorname{grad}\varphi_i$ and $\operatorname{grad}\varphi_j$ have common support (the region on which the function is non-zero) only on the triangles incident to both i and j.

(a) Show that the aspect ratio of a triangle can be expressed as the sum of the cotangents of the interior angles at its base, *i.e.*,

$$\frac{w}{h} = \cot\alpha + \cot\beta.$$



(b) Show that the gradient of the hat function on triangle t_{ijk} is given by

$$\operatorname{grad}\varphi_{\mathsf{i}} = \frac{e_{\mathsf{jk}}^{\perp}}{2A_{\mathsf{ijk}}}$$

where $e_{\mathsf{jk}}^{\perp}$ is the vector e_{jk} rotated 90° counterclockwise within t_{ijk}.

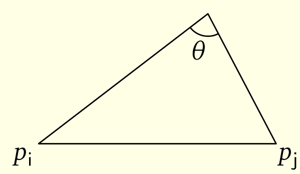
(c) Show that for any hat function φ_{i} associated with vertex p_{i} of triangle t_{ijk},

$$\langle \operatorname{grad}\varphi_{\mathsf{i}}, \operatorname{grad}\varphi_{\mathsf{i}}\rangle A_{\mathsf{ijk}} = \tfrac{1}{2}(\cot\alpha + \cot\beta).$$

(d) Show that for the hat functions φ_{i} and φ_{j} associated with vertices p_{i} and p_{j} of triangle t_{ijk}, we have

$$\langle \operatorname{grad}\varphi_{\mathsf{i}}, \operatorname{grad}\varphi_{\mathsf{j}}\rangle A_{\mathsf{ijk}} = -\tfrac{1}{2}\cot\theta$$

where θ is the angle between the opposite edge vectors.



Putting all these facts together, we have the infamous *cotan formula*

$$(Lu)_{\mathsf{i}} = \tfrac{1}{2}\sum_{e_{\mathsf{ij}}\succ p_{\mathsf{i}}}(\cot\alpha_{\mathsf{ij}} + \cot\beta_{\mathsf{ij}})(u_{\mathsf{j}} - u_{\mathsf{i}}).$$

where α_{ij} and β_{ij} are the angles of opposite vertices across from e_{ij} in the two adjacent triangles.

On a discrete surface, it happens that

$$L = -d_0^{\mathsf{T}} *_1 d_0 .$$

Since the finite element mass matrix $*_0^{\mathrm{FEM}}$ is not a diagonal matrix, it does not agree with the diagonal discrete Hodge star $*_0^{\mathrm{diag}}$. One can show that $*_0^{\mathrm{diag}}$ and $*_0^{\mathrm{FEM}}$ has the same row sum. The diagonal entries of $*_0^{\mathrm{diag}}$ is often referred as the **lumped mass**, and it has been shown that replacing $*_0^{\mathrm{FEM}}$ with $*_0^{\mathrm{diag}}$ does not affect the order of accuracy.

5.1.6 Time discretization for the heat flow

So far we have seen the heat equation discretized in space, but is still continuous in the time variable:

$$*_0 \tfrac{d}{dt} u = Lu, \quad u \in C^0(M;\mathbb{R}), \quad L = -d_0^{\intercal} *_1 d_0 \,.$$

One sure can plug this system ODE to any ODE solver. Here we note on a few popular ODE schemes and comment on their stability.

Denote $0 = t_0 < t_1 < t_2 < \ldots$ be the discretization of the time $\mathbb{R}^+ = \{0 \leq t < \infty\}$. Assume uniform time step $t_n - t_{n-1} = \Delta t$. Let $u^{(n)} \in C^0(M;\mathbb{R})$ be the approximated solution at $t = t_n$.

One of the most straightforward schemes is the **forward Euler scheme**, which directly replaces $\frac{d}{dt}$ by the forward difference:

$$*_0 \tfrac{1}{\Delta t}\left(u^{(n+1)} - u^{(n)}\right) = Lu^{(n)};$$

after rearrangement,

$$u^{(n+1)} = u^{(n)} + \Delta t *_0^{-1} Lu^{(n)}. \tag{5.10}$$

This is an **explicit** iteration: one may update $u^{(n+1)}$ directly from $u^{(n)}$ by operations involving at most matrix multiplication. (When $*_0$ is not diagonal, you may need to solve a linear system with a symmetric positive definite matrix $*_0$).

The **backward Euler scheme** is similar but **implicit**:

$$*_0 \tfrac{1}{\Delta t}\left(u^{(n+1)} - u^{(n)}\right) = Lu^{(n+1)};$$

after rearrangement,

$$(*_0 - \Delta t L)\, u^{(n+1)} = *_0 u^{(n)}. \tag{5.11}$$

To calculate $u^{(n+1)}$ from a current $u^{(n)}$, one has to solve a linear system with a symmetric positive definite matrix $(*_0 - \Delta t L)$. (Recall $L = -d_0^{\intercal} *_1 d_0$, which is symmetric negative semi-definite).

Both the forward and backward Euler time stepping are first-order accurate in time. However, in terms of stability, the forward Euler scheme is unstable unless Δt is extremely small; on the other hand the backward Euler scheme is unconditionally stable (stability is independent of the choice of Δt).

Stability

The most elementary way to analyze stability is by the **von Neumann stability analysis**, where one investigates the stability of the time discretization for each eigenvector of the Laplacian. Let λ_k, v_k be the eigenvalues and eigenvectors of the (generalized) eigenvalue problem

$$-Lv_k = \lambda_k *_0 v_k$$

for $k = 1, \ldots, N$ when L is N-by-N (*i.e.* there are N vertices). Note that $\lambda_k \geq 0$ since L is negative semi-definite.

Now, plugging a single eigen-mode $u^{(n)} = c_k^{(n)} v_k$ (where $c_k^{(n)}$ is the time-dependent amplitude) into the forward Euler scheme (5.10), we obtain a difference equation

$$c_k^{(n+1)} = (1 - \Delta t \lambda_k) c_k^{(n)}.$$

This first order difference equation (yielding a geometric progression) is stable *only when* $|1 - \Delta t \lambda_k| \leq 1$. This implies that stability requires

$$\Delta t \leq \frac{2}{\lambda_k}$$

for all eigenvalues λ_k. A fact about the eigenvalues $-Lv_k = \lambda_k *_0 v_k$ is that the maximum eigenvalue λ_N is of order $\frac{1}{\ell^2}$ where ℓ is a typical edge length. That is, in order to simulate the heat equation using the forward Euler method, you have to pick

$$\Delta t \leq O(\ell^2)$$

otherwise the solution will blow up on you.

Using the same technique let us investigate the backward Euler method. Plugging in a single eigen-mode $u^{(n)} = c_k^{(n)} v_k$, (5.11) reads

$$c_k^{(n+1)} = \frac{1}{1 + \Delta t \lambda_k} c_k^{(n)}.$$

Note that for any $\Delta t > 0$ the amplification factor $\frac{1}{1+\Delta t \lambda_k} \leq 1$ and $= 1$ only when $\lambda = 0$. (One needs to check the eigenspace for $\lambda = 0$ is at most 1-dimensional). The unconditional stability follows.

Exercise 5.6 (Unconditional instability)**.** The **leapfrog scheme** for the heat equation is given by

$$*_0 \frac{1}{2\Delta t} \left(u^{(n+1)} - u^{(n-1)} \right) = Lu^{(n)}$$

which allows one to calculate $u^{(n+1)}$ using $u^{(n-1)}$ and $u^{(n)}$ with second order accuracy in time. Show that the leapfrog scheme for the heat equation is however *unstable* for *any*

$\Delta t > 0$.

Hint: A linear second order difference equation $a_2 c^{(n+2)} + a_1 c^{(n+1)} + a_0 c^{(n)} = 0$ *is unstable if there is a root* $z \in \mathbb{C}$ *of the polynomial* $a_2 x^2 + a_1 x + a_0$ *that* $|z| > 1$.

Exercise 5.7 (Crank-Nicolson)**.** The **Crank-Nicolson method** for the heat equation is given by

$$*_0 \tfrac{1}{\Delta t}\left(u^{(n+1)} - u^{(n)}\right) = L\frac{u^{(n)} + u^{(n+1)}}{2},$$

which is an implicit method with second order accuracy in time. Show that it is stable for all choice of $\Delta t > 0$ (*i.e.* unconditionally stable).

5.2 Poisson Equation

On a manifold M, given a function $f \in \Omega^0(M;\mathbb{R})$, the **Poisson equation** for $u \in \Omega^0(M;\mathbb{R})$ is given by

$$\Delta u = f. \tag{5.12}$$

You can think of it as the steady solution of the heat equation

$$\tfrac{\partial}{\partial t} u = \Delta u + *_0^{-1}\beta$$

with $\frac{\partial}{\partial t} u = 0$ and $\beta = -*_0 f$. For $\partial M = \emptyset$, the weak formulation of the Poisson equation is that

$$-\langle\!\langle d\varphi, du\rangle\!\rangle = \langle\!\langle \varphi, f\rangle\!\rangle$$

for all test functions $\varphi \in \Omega^0(M;\mathbb{R})$. The solution to (5.12) is also the minimizer to the energy

$$E(u) = \tfrac{1}{2}\|du\|^2 + \langle\!\langle u, f\rangle\!\rangle.$$

Think of it as the final destination of a heat flow as the gradient flow for this convex functional.

The discrete Poisson equation is written down straightforward by DEC: given $f \in C^0(M;\mathbb{R})$, solve for $u \in C^0(M;\mathbb{R})$ satisfying

$$-d_0^{\intercal} *_1 d_0\, u = *_0 f \quad \text{or simply} \quad Lu = *_0 f.$$

Notice that we write $d_0^{\intercal} *_1 d_0$ on the LHS instead of $*_0^{-1} d_0^{\intercal} *_1 d_0$ in order to obtain a linear system with a symmetric system matrix.

5.2.1 Fredholm alternative

The equation (5.12) should be thought of as a linear algebra problem like $Ax = b$. A natural question to ask here is that, which choices of b admit the existence of the solution x? To answer this type of questions we recall the following basic linear algebra fact.

Lemma 5.1. *Let $A\colon V \to W$ be a linear operator from space V to W. Let $A^*\colon W \to V$ be the adjoint of A, which is the operator that satisfies $\langle Ax, y\rangle_W = \langle x, A^* y\rangle_V$ for all $x \in V$ and $y \in W$. Then $\ker(A)^\perp = \operatorname{im}(A^*)$, and $\operatorname{im}(A)^\perp = \ker(A^*)$.*

Back to the Poisson equation. On a manifold M without boundary, we have that Δ is self-adjoint ($\Delta = \Delta^*$) since $\langle\!\langle \Delta u, v\rangle\!\rangle = -\langle\!\langle du, dv\rangle\!\rangle = \langle\!\langle u, \Delta v\rangle\!\rangle$. With the following exercise, you will show that $\ker(\Delta) = \ker(\Delta^*)$ consists of functions constant on connected component of M.

Exercise 5.8 (Harmonic functions on M)**.** Suppose M has no boundary. Show that $\Delta u = 0$ if and only if u is constant on each connected component of M.

Hint: Show $\|du\|^2 = 0$.

The Poisson equation (5.12) admits a solution if and only if $f \in \operatorname{im}(\Delta)$. Knowing $\ker(\Delta)$ we conclude the following theorem.

Theorem 5.2. *Suppose M is a manifold without boundary, and $M = M_1 \sqcup \ldots \sqcup M_m$ where each M_k is a connected component of M. Then*

$$\Delta u = f$$

has a solution if and only if $\int_{M_k} *f = 0$ for each $k = 1, \ldots, m$.

In most cases the domain M is connected. In this case $\Delta u = f$ has a solution if and only if f has zero mean.

How does this translate into the discrete setting? In the discrete setting one solves the linear system

$$Lu = *_0 f$$

where $u \in C^0(M; \mathbb{R})$ and the matrix $L = -d_0^\intercal *_1 d_0$. Assume that the diagonal Hodge star $*_1$ is a diagonal matrix with positive diagonal entries. Then the only kernel of L is the subspace spanned by constant function (assuming one connected component). That is, the right-hand side array $*_0 f$ must have zero sum of entries; in other words, zero sum of f weighted by point area encoded in $*_0$ analogous to $\int_M *f = 0$.

The Poisson equation $\Delta u = f$ (and so is the discrete $Lu = *_0 f$) is not well-posed if f has constant component. In some situation it makes sense to remove the constant component. That is, instead of trying to solve $\Delta u = f$ one can solve $\Delta u = f - \overline{f}$ where $\overline{f} = \frac{1}{\int_M *1} \int_M *f$.

Similarly in the discrete version where one attempts to solve $Lu = b$ with the right-hand side array $b = *_0 f$ being area weighted values of f on vertices, one can instead solve $Lu = b - \overline{b}$ where $\overline{b} = \frac{1}{n}\sum_{\mathrm{i}=0}^{n-1} b_{\mathrm{i}}$ and n is the number of vertices. Of course you must be certain that this trick makes sense in the context of your particular problem.

5.3 A Note on Implementation

What we have seen so far are the heat equation

$$\frac{\partial}{\partial t} u = \Delta u$$

and the Poisson equation

$$\Delta u = f$$

for $u \in \Omega^0(M;\mathbb{R})$ on closed manifold M. We have also seen their discrete versions. A suitable exercise here would be implementing a Poisson equation solver on a closed surface. As it turns out, there are many differential equations that boil down to solving Poisson-like equations as subproblem; backward Euler scheme for heat equation (5.11) is one example.

As we found out earlier, the Laplacian of a 0-form u is approximated by

$$(Lu)_{\mathrm{i}} = \frac{1}{2} \sum_{e_{\mathrm{ij}} \succ p_{\mathrm{i}}} \left(\cot \alpha_{\mathrm{ij}} + \cot \beta_{\mathrm{ij}}\right)(u_{\mathrm{j}} - u_{\mathrm{i}})$$

where α_{ij} and β_{ij} are the angles of opposite vertices across from e_{ij} in the two adjacent triangles. In terms of matrix element this is the same as saying

$$\begin{aligned} L_{\mathrm{ij}} &= \frac{1}{2}\left(\cot \alpha_{\mathrm{ij}} + \cot \beta_{\mathrm{ij}}\right) \quad \text{when i, j is connected by an edge} \\ L_{\mathrm{ii}} &= -\frac{1}{2} \sum_{e_{\mathrm{ij}} \succ p_{\mathrm{i}}} \left(\cot \alpha_{\mathrm{ij}} + \cot \beta_{\mathrm{ij}}\right) \end{aligned}$$

and all other matrix elements are zero.

A straightforward way to construct L in the code is visit all half-edges $e_{\mathrm{ij}}^{\mathrm{k}}$, where $e_{\mathrm{ij}}^{\mathrm{k}}$ is the edge in triangle t_{ijk} across from vertex p_{k}, and add entries L_{ij}, L_{ji} with $\frac{1}{2}\cot\theta_{\mathrm{ij}}^{\mathrm{k}}$, and subtract $-\frac{1}{2}\cot\theta_{\mathrm{ij}}^{\mathrm{k}}$ from L_{ii} and L_{jj}. Here $\theta_{\mathrm{ij}}^{\mathrm{k}}$ is the angle in t_{ijk} at vertex k. See Algorithm 1.

However, most sparse matrix library would prefer to construct the matrix all at once rather than first creating an empty sparse matrix and then jumping around the entries modifying its values. These sparse matrix constructors usually take in five informations: R array of row indices, C array of column indices, V array of entry values, m, n size of matrix. Effectively the constructed matrix A is m-by-n with value V_k added to the (R_k, C_k) entry of A.

Algorithm 1 Build cotan Laplace (assign entries)

L sparse $n \times n$ zero matrix ▷ n is the number of points
for each half-edge $e_{\mathrm{ij}}^{\mathrm{k}}$ **do**
  $w := \frac{1}{2}\cot\theta_{\mathrm{ij}}^{\mathrm{k}}$ ▷ $\theta_{\mathrm{ij}}^{\mathrm{k}}$ is the angle opposite to the half-edge $e_{\mathrm{ij}}^{\mathrm{k}}$.
  $L_{\mathrm{ij}} += w,\ L_{\mathrm{ji}} += w$
  $L_{\mathrm{ii}} -= w,\ L_{\mathrm{jj}} -= w$
end for

If you are familiar with MATLAB, it is the function call

```
sparse(R,C,V,m,n)
```

In Python, one loads the scipy package of Python. One of the methods for constructing sparse matrix is given by

```
scipy.sparse.csr_matrix((V,(R,C)),shape=(m,n))
```

Here `scipy.sparse.csr_matrix` constructs "Compressed Sparse Row" matrix. Alternatively one may use `scipy.sparse.csc_matrix` (Compressed Sparse Column).

Therefore, in practice, one can build the sparse Laplace matrix given by Algorithm 2.

Algorithm 2 Build cotan Laplace (construct sparse matrix all at once)

I, J, W are $m \times 1$ empty (zero) array ▷ m is the number of half edges
for $k = 0, \ldots, m-1$ **do**
  Let h_k be the k-th half-edge
  Let i, j be the indices of the source and destination points of h_k
  Let $\cot\theta$ be the cotan of the angle opposite to h_k.
  $I(k) \leftarrow \mathrm{i}, J(k) \leftarrow \mathrm{j}, W(k) \leftarrow \frac{1}{2}\cot\theta$.
end for

$$R = \begin{bmatrix} I \\ I \\ J \\ J \end{bmatrix}, \quad C = \begin{bmatrix} I \\ J \\ I \\ J \end{bmatrix}, \quad V = \begin{bmatrix} -W \\ W \\ W \\ -W \end{bmatrix}$$ ▷ The rows, columns and values of L.

$\mathrm{L} = \mathrm{Sparse}(R, C, V, n, n)$ ▷ The constructor of a sparse matrix.
▷ (R, C, V, n, n) specifies the rows, columns, values and the size of the matrix.

Exercise 5.9 (Scalar Poisson Equation)**.** In this coding exercise you will build the cotan-Laplace matrix for solving the discrete Poisson equation

$$Lu = *_0 f$$

on a triangular mesh, as an approximation to the Poisson equation

$$\Delta u = *_0^{-1} d * du = f.$$

You can imagine f is the electric charge density as a function, and u is the electric potential for $E = -du$ being the electric field (for more detail see Section 5.4 and assume $\frac{\partial}{\partial t} B = 0$ and E is the d of some 0-form). You can also imagine f is the heat source as a function (or $*_0 f$ the energy per area per second) in a steady heat equation.

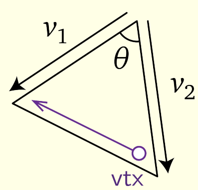
(a) Recall that **Houdini**'s vertices are polygon corners that 1-to-1 corresponds to half-edges through VEX functions

`vertexhedge` and `hedge_srcvertex` .

We use this correspondence to store the quantities of half-edges as vertex attributes. For example,

```
// Vertex wrangle
i@ind = i@vtxnum;                    // ind is the vertex index
int he = vertexhedge(0,@vtxnum);
i@src = hedge_srcpoint(0,he);        // source point of the half-edge
i@dst = hedge_dstpoint(0,he);        // destination of the half-edge
i@opp = hedge_presrcpoint(0,he);// opposite point across from the
                                     // halfedge
```

Derive an expression for the cotangent of a given angle purely in terms of the two incident edge vectors and the standard Euclidean dot product and cross product.



Implement a Vertex Wrangle node to create and define vertex attribute

`f@cotan`

(b) Each triangle has an area and each point has a dual area. For dual area you can simply use 1/3 the total area of the incident faces:

$$A_{\mathrm{i}} = \frac{1}{3} \sum_{t_{\mathrm{ijk}} \succ p_{\mathrm{i}}} A_{\mathrm{ijk}}.$$

Note that 1/3-total-area is just an approximation as if the dual vertices are the barycenters of the triangles. (What would be the correct circumcenter point area?) Create and calculate the value of point attribute `f@area` using VEX programing. (You can also use Measure and Attribute Promote if you are sure the outcome is what you want).

(c) Place a Python node and name the node "Laplacian". In it build the Laplace matrix L and the mass matrix M (*i.e.* $*_0$ matrix) by completing the following Python script. *Hint: Algorithm 1 or Algorithm 2.*

```
# IMPORT PACKAGES
import numpy as np
import scipy.sparse as sp

node = hou.pwd()
geo = node.geometry()

# Read number of points and halfedges (vertices)
n_pt = geo.intrinsicValue("pointcount")
n_vtx = geo.intrinsicValue("vertexcount")

# Loop over vertices
for prim in geo.prims():
    for vtx in prim.vertices():
        # Examples of evaluating vtx attributes
        ind   = vtx.attribValue("ind")
        src   = vtx.attribValue("src")
        dst   = vtx.attribValue("dst")
        cotan = vtx.attribValue("cotan")
        # TODO
        # ...


# Build Laplacian
# L = ... TODO

# Build Mass (no need to do further coding here)
# The function hou.Geometry.pointFloatAttribValues quickly
# returns the values of point attributes from all points
# as a tuple list.  Numpy's array converts it to a numpy array
# which can be fed into spdiags (sparse diagonal matrix)
ptarea = np.array(geo.pointFloatAttribValues("area"))
M = sp.spdiags(ptarea,0,n_pt,n_pt)

# Cache result
#node.setCachedUserData("L",L)   #<- TO REMOVE COMMENT
node.setCachedUserData("M",M)
```

(d) In this exercise we let you explore a few simple things you can test/visualize the matrix you have just constructed. Now you have cached sparse matrices `"L"` and `"M"` in the node "Laplacian". You can re-access these sparse matrices from another Python node by calling

```
the_laplacian_node = hou.node("../Laplacian")
L = the_lapacian_node.cachedUserData("L")
M = the_lapacian_node.cachedUserData("M")
```

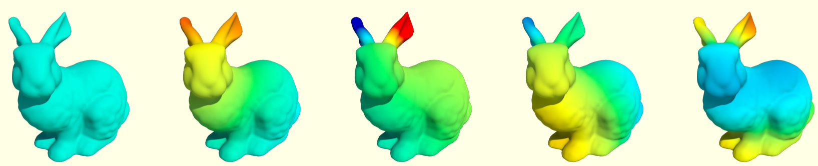
Now you can do a few things to test if the Laplace matrix is correct. First of all, the Laplacian eigenfunction should be the "standing waves" of this domain. A function φ is an eigenfunction of Laplacian if there is $\lambda \in \mathbb{R}$ so that

$$(-\Delta)\varphi = \lambda\varphi.$$

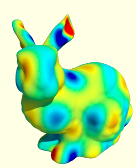
Eigenmodes φ and λ are approximated by their discrete counterpart, in which case one solves the generalized eigenvalue problem

$$-L\varphi = \lambda *_0 \varphi.$$

Here λ should be all greater or equal to zero. The number of zero eigenvalues is the dimension of the null space of Δ, which you know from Exercise 5.8 that it is the number of connected components. If the domain is a unit sphere, the first few eigenvalue/eigenfunction should well-approximate the **spherical harmonics**. If you have been using Laplacian and Stanford Bunny a lot, you will remember that the first few eigenfunctions on the bunny are



and the 99th eigenfunction looks like



Here is a sample Python code for eigenvalue problem

```
import numpy as np
import scipy.sparse as sp
import scipy.sparse.linalg as la

# current node object and geometry object
node = hou.pwd()
geo = node.geometry()

LaplacianNode = hou.node("../Laplacian")
L = LaplacianNode.cachedUserData("L")
```

```
    M = LaplacianNode.cachedUserData("M")

    (ew,ev) = la.eigsh(-L,k=100,M=M,sigma=0.1)

    geo.setPointFloatAttribValues("phi",ev[:,99])
    # make sure you have defined a point attribute called phi.
```

(e) Now solve the Poisson equation

$$-Lu = *_0 f.$$

You can set $\int_{*p_i} *_0 f = \pm 1$ on a few points as point charges, and zero elsewhere. Taking advantage of user interface, you can use a Point node, on the right of the "Group" entry click on to manually select points from the Scene View, and then assign values to the selected points.
Place another Python node where you solve

$$-Lu = *_0 f$$

using functions in `scipy.sparse.linalg` of the Scipy library. Iterative linear solvers are recommended. Since L is not invertible (it has kernel) you may want to use least-squares solvers.
By visualizing the solution u (the color) and its gradient (the field line), you obtain the electric potential and electric field for given point charges on a curved surface:

5.4 Classical Maxwell's Equations

The classical Maxwell's equations describe the time evolution of the electric field (1-form) and magnetic field (2-form) interacting with electrically charged matter. In this section we consider the domain being a manifold M representing the space, with time being a separate dimension. In the next section we discuss the relativistic electromagnetism, in which case we consider the domain M being the space and time together as a manifold called a spacetime.

5.4.1 Charge and current

Let M be an n-dimensional manifold. The **electric charge density** is a (coulomb-valued) n-form

$$\rho \in \Omega^n(M; \mathbb{R}\,\mathrm{C})$$

whose sign (compared with the ambient orientation $*1$) represent the sign of the charge. Here $\mathbb{R}\,\mathrm{C}$ is the real numbers with unit *coulomb*. On a region $V \subset M$, $\int_V \rho$ represent the total charge in the volume V. The **electric current** j is a (coulomb-per-second-valued) $(n-1)$-form

$$j \in \Omega^{n-1}(M; \mathbb{R}\,\mathrm{C/s})$$

describing the rate electric charge passing through a section per unit area per time. The **conservation of electric charge** states that

$$\tfrac{\partial}{\partial t}\rho + dj = 0. \tag{5.13}$$

Integrating (5.13) over a volume, using Stokes' Theorem, one has

$$\frac{d}{dt}\int_V \rho = -\int_{\partial V} j$$

which reads that the change of total charge is contributed only from the in-flowing current.

5.4.2 Electric and magnetic field

To introduce the electric field and magnetic field, we consider a single point mass (particle) of charge q located at $p \in M$ and moving with velocity $v \in T_pM$. A force applied on the particle is written as an *energy*-valued 1-form $f \in \Omega^1_p(M; \mathbb{R}\,\mathrm{J})$. (In particular the unit $[f]$ is Newton and $\int_\gamma f$ along a particle path γ is the total work.) The force due to electromagnetism is given by the **Lorentz force law**

$$f = q(E - \iota_v B).$$

Here ι_v is the contraction operator (Appendix 5.A). By measuring f at every $p \in M$ and $v \in T_pM$ we determine a 1-form $E \in \Omega^1(M; \mathbb{R}\,\mathrm{J/C})$ called the **electric field**, and a 2-form $B \in \Omega^2(M; \mathbb{R}\,\mathrm{J{\cdot}s/C})$ called the **magnetic field**.

The Lorentz force law for moving charged particles can be generalized for charged fluids. Let $\rho \in \Omega^n(M; \mathbb{R}\,\mathrm{C})$ be the charge density, which flows according to a velocity vector field $v \in \Gamma(TM)$. Then the force field $f \in \Omega^1(M; \mathbb{R}\,\mathrm{J})$ on the fluid is given by

$$f = (*_0^{-1}\rho)(E - \iota_v B).$$

5.4.3 Maxwell's Equations

The (classical) Maxwell's equation is the differential equation for the electric field and magnetic field in the presence of charged density and current and time variation of the fields themselves. The classical Maxwell's equations consist of the four equations: the absence of magnetic charge, Faraday's law, Gauss' law, and Ampere-Maxwell law.

The absence of magnetic charge

This law states that $B \in \Omega^2(M; \mathbb{R}\,\mathrm{J{\cdot}s/C})$ is **closed**, *i.e.*

$$dB = 0. \tag{5.14}$$

When $M = \mathbb{R}^3$ and write the magnetic field as a vector $\vec{\mathbf{B}} = (*B)^\sharp$, then $dB = 0$ amounts to the divergence free condition $\nabla \cdot \vec{\mathbf{B}} = 0$.

Faraday's law

Micheal Faraday discovered that a time variation in a magnetic field induces a circulation of electric field

$$dE = -\tfrac{\partial}{\partial t} B. \tag{5.15}$$

The above two equations (5.14) and (5.15) do not require metric or $*$. They are just topological properties. The remaining equations require Hodge stars, which require either a metric on M, or some general weights as the electric permittivity and magnetic permeability as the Hodge stars.

Gauss' law

Gauss' law states that

$$d{*}E = \rho. \tag{5.16}$$

To understand this equation, take the integral of it over a volume V:

$$\int_{\partial V} {*}E = \int_V \rho$$

which reads that the total flux of electric field over the boundary surface is related to the total charge enclosed. Here the Hodge star $* = *_1$ is regarded as a map from primal 1-forms to dual $(n-1)$-forms

$$*_1 \colon \Omega^1(M; \mathbb{R}\,{}^{\mathrm{J}}\!/\!_{\mathrm{C}}) \to \Omega^{n-1}(M; \mathbb{R}\,\mathrm{C})$$

encoding the physical relation called the **electric permittivity** (often denoted with ϵ). In the vacuum in $\mathbb{R}^3$ without distinguishing primal and dual forms, one has $*_1\, dx = \epsilon_0\, dy \wedge dz$ with $\epsilon_0 \approx 8.85 \times 10^{-12}\, \mathrm{C^2/N{\cdot}m^2}$. In a general material, at each point $p \in M$ the permittivity $*_1$ is a general linear map from $\Omega^1(M; \mathbb{R}\,\mathrm{J/C})$ to $\Omega^{n-1}(M; \mathbb{R}\,\mathrm{C})$. The form $D = {*}E$ is also known as **electric displacement field**.

Ampere-Maxwell law

Ampere-Maxwell law states that

$$d{*_2}B = j + *_1 \tfrac{\partial}{\partial t} E. \tag{5.17}$$

Here

$$*_2 \colon \Omega^2(M; \mathbb{R}\,\mathrm{J{\cdot}s/C}) \to \Omega^{n-2}(M; \mathbb{R}\,\mathrm{C/s})$$

is called the **magnetic permeability** often denoted as $\frac{1}{\mu}$. In the vacuum $\mathbb{R}^3$, $*_2\, dx \wedge dy = \frac{1}{\mu_0}\, dz$ where $\mu_0 \approx 1.26 \times 10^{-6}\, \mathrm{N{\cdot}s^2/C^2}$. The $(n-2)$-form $H = *_2 B \in \Omega^{n-2}(M; \mathbb{R}\,\mathrm{C/s})$ is also known as the **auxiliary magnetic field**. Note that $\epsilon_0 \mu_0 = \frac{1}{c^2}$ where c is the vacuum speed of light (as would be seen later from the electromagnetic wave section.)

A summary

In summary, the classical Maxwell's equations are given by the following. Given a permittivity $*_1 \colon \Omega^1(M; \mathbb{R}\,\mathrm{J/C}) \to \Omega^{n-1}(M; \mathbb{R}\,\mathrm{C})$, a permeability $*_2 \colon \Omega^2(M; \mathbb{R}\,\mathrm{J{\cdot}s/C}) \to \Omega^{n-2}(M; \mathbb{R}\,\mathrm{C/s})$, and a charge density $\rho(t) \in \Omega^n(M;\mathbb{R}\,\mathrm{C})$ and current $j(t) \in \Omega^{n-1}(M;\mathbb{R}\,\mathrm{C})$, then the electric field $E \in \Omega^1(M;\mathbb{R}\,\mathrm{J/C})$ and the magnetic field $B \in \Omega^1(M;\mathbb{R}\,\mathrm{J{\cdot}s/C})$ satisfy

$$\begin{aligned} d *_1 E &= \rho \\ dB &= 0 \\ dE &= -\tfrac{\partial}{\partial t} B \\ d *_2 B &= j + *_1 \tfrac{\partial}{\partial t} E. \end{aligned}$$

5.4.4 Electromagnetic wave

In the absence of ρ and j, *i.e.* with no free charge and free current (not necessarily vacuum), the electromagnetic fields E, M inducts each other and form electromagnetic waves.

Take a time derivative to Ampere-Maxwell law (5.17), which gives

$$d *_2 \tfrac{\partial}{\partial t} B = *_1 \tfrac{\partial^2}{\partial t^2} E.$$

Then replace $\frac{\partial}{\partial t} B$ by $-dE$ using Faraday's law (5.15), and obtain

$$-d *_2 dE = \tfrac{\partial}{\partial t} j + *_1 \tfrac{\partial^2}{\partial t^2} E. \tag{5.18}$$

Recall that the codifferential

$$\delta_k = (-1)^k (*_{k-1})^{-1} \, d_{n-k} *_k$$

and the Laplacian is given by

$$\Delta_k = -d_{k-1}\,\delta_k - \delta_{k+1}\,d_k\,.$$

After rearranging (5.18) we have

$$\tfrac{\partial^2}{\partial t^2} E = - *_1^{-1} d *_2 dE = -\delta\, dE.$$

From the Gauss' law (5.16) $d *_1 E = \rho$ with $\rho = 0$, we have that $\delta E = - *_0^{-1} d *_1 E = 0$. Therefore $d\,\delta E = 0$ and $\Delta E = (-d\,\delta - \delta\,d)E = -\delta\,dE$. Therefore

$$\tfrac{\partial^2}{\partial t^2} E = \Delta E. \tag{5.19}$$

This is the **wave equation** for an 1-form E.

Exercise 5.10. In the absence of ρ and j, show that B satisfies

$$\frac{\partial^2}{\partial t^2}B = \Delta B = d *_1^{-1} d *_2 B. \tag{5.20}$$

The **wave speed** of the wave equations (5.19) and (5.20) are encoded by Δ_1 and Δ_2 respectively. Namely, in both cases the wave speed is determined by the product of the factor contributed from $*_1^{-1}$ and $*_2$. In the $\mathbb{R}^3$ vacuum $*_1 = \epsilon_0 *_1^{\mathbb{R}^3}$ and $*_2 = \frac{1}{\mu_0} *_2^{\mathbb{R}^3}$, where $*^{\mathbb{R}^3}$ is the usual dimensionless Hodge star induced from the standard metric of $\mathbb{R}^3$. So $\Delta_1 = c^2 \Delta_1^{\mathbb{R}^3}$ and $\Delta_2 = c^2 \Delta_2^{\mathbb{R}^3}$ where $c^2 = \frac{1}{\epsilon_0 \mu_0}$. In a more general material one has a more general $*_1$ and $*_2$ that steer the propagation of electromagnetic waves.

5.5 Relativistic Maxwell's Equations

In the previous section, the electromagnetism is formulated with time-dependent 1-form E and 2-form B on a manifold M where M is usually 3-dimensional which represents space. In this section, we consider M as a 4-dimensional manifold including the information of space and time, and the electric and magnetic fields are united as 2-form defined on spacetime.

5.5.1 Electromagnetism on $\mathbb{R} \times M$

Before going to the full relativistic version, let us look at a simpler setup. A straightforward way of putting both time and space together as a domain manifold is by taking the Cartesian product

$$\widetilde{M} := \mathbb{R} \times M$$

where the $\mathbb{R}$ component represents the time, and the n-dimensional (usually 3-dimensional) manifold M represents the space. Let t be the coordinate of the time component. Then $dt \in \Omega^1(\widetilde{M}; \mathbb{R}\mathrm{s})$ is a natural time-valued 1-form. Its corresponding vector field $dt^\sharp$ is the time direction in $\widetilde{M}$.

Note that this setup is only a quick way to build a spacetime $\widetilde{M}$. In the theory of relativity, there is no preferred time direction, so there is no canonical way to decompose $\widetilde{M}$ into $\mathbb{R} \times M$.

On $\widetilde{M}$ we introduce a pseudo-metric so that

$$\begin{aligned} \langle dt^\sharp, dt^\sharp \rangle &= -c^2, \\ \langle X, X \rangle &= \langle X, X \rangle_M, \quad \text{for } X \in T_pM \subset T_{(t,p)}\widetilde{M} \end{aligned}$$

where $\langle .,. \rangle_M$ is the metric from M. The reason for this choice of metric will explained in the next subsection on special relativity. We also have a volume form for $\widetilde{M}$. If μ_M is the volume n-form for M, then the volume $(n+1)$-form μ for $\widetilde{M}$ is given by

$$\mu = dt \wedge \mu_M.$$

Suppose $*^M$ is the Hodge star for M, then the Hodge star $*$ for $\widetilde{M}$ is given by the following. If ω is a k-form *tangent to* M, *i.e.* $\omega(dt^\sharp) = 0$, then

$$*\omega = (-1)^k \, dt \wedge *_M \omega, \quad *(dt \wedge \omega) = -c^2 *_M \omega.$$

One should check $\alpha \wedge *\alpha = \langle \alpha, \alpha \rangle \mu$.

Now, electric charge density $\rho \in \Omega^n(M; \mathbb{R}\,\mathrm{C})$ and flux $j \in \Omega^{n-1}(M; \mathbb{R}\,\mathrm{C/s})$ can be unified as the **4-current** (4 is the case when $n = 3$ and $\widetilde{M}$ is 4-dimensional)

$$J := \rho - dt \wedge j \in \Omega^n(\widetilde{M}; \mathbb{R}\,\mathrm{C}).$$

The conservation of charge $\frac{\partial}{\partial t}\rho + d_M j = 0$ is then equivalent to

$$dJ = 0.$$

One checks that

$$dJ = dt \wedge \tfrac{\partial}{\partial t}\rho + dt \wedge d_M j = 0$$

where d_M is the exterior derivative on M.

The electric field and magnetic field on M can be unified as a 2-form called the **Faraday 2-form** defined on $\widetilde{M}$:

$$F := -dt \wedge E + B \in \Omega^2(M; \mathbb{R}\,\mathrm{J{\cdot}s/C}).$$

Suppose at time t_0 there is a moving charged particle at $p \in M$ with velocity $v_M \in T_p M$. Then the particle is located at $(t_0, p) \in \widetilde{M}$ with velocity $v = dt^\sharp + v_M \in T_{(t_0,p)}\widetilde{M}$. Suppose the particle has electric charge q. Then its Lorentz force is given by

$$f = -q \iota_v F.$$

Finally, the Maxwell's equations in terms of the Faraday 2-form is concisely written as

$$\begin{cases} dF = 0 \\ d{*F} = J. \end{cases} \tag{5.21}$$

The first equation $dF = 0$ in expansion is

$$dt \wedge d_M E + dt \wedge \tfrac{\partial}{\partial t} B + d_M B = 0$$

which unifies $d_M B = 0$ (by taking another $dt\wedge$ to the equation) and the Faraday's Law $d_M E + \frac{\partial}{\partial t} B = 0$ (by applying $\iota_{dt^\sharp}$ to the equation). The second equation $J = d{*}F$ is

$$\begin{aligned}\rho - dt \wedge j &= d(-c^2 *_M E + dt \wedge *_M B) \\ &= -c^2 d_M *_M E - c^2\, dt \wedge *_M \tfrac{\partial}{\partial t} E - dt \wedge d_M *_M B\end{aligned}$$

unifying the Gauss' Law and Ampere-Maxwell's Law.

One can almost immediately write down the corresponding DEC formulation for the discrete version of (5.21). The corresponding algorithm for the case when the mesh being regular was also known as the **Finite Difference Time Domain (FDTD)** method, or **Yee's Scheme**. DEC approach generalizes Yee's Scheme to arbitrary mesh for $\widetilde{M}$, even with non-synchronous time grid (EM-fields are solved at different time resolution in different cells).

5.5.2 Special Relativity

In order to understand the spacetime domain on which we are going to define relativistic electromagnetic fields independent of the choice of time-axis, we need a bit background on Relativity. In particular Special Relativity provides a model for space and time.

Albert Einstein postulated in one of his 1905 papers that

1. the laws of physics are the same in all **inertial frames of reference**.
2. the speed of light in free space has the same value c in all inertial frames of reference.

The first postulate says that if a physical law is formulated in terms of a coordinate system (t, x, y, z), then it must take the same form when written in terms of another coordinate system $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ which is a uniform translating motion with respect to (t, x, y, z). This postulate has been known since Galileo Galilei and is applicable to all classical physics. The second postulate is more counterintuitive in the classical sense, and it is just based on the surprising result of 1887 Michelson-Morley experiment: the measurement of the speed of light is the same in all seasons (when earth is moving in different directions!)

To have the two postulates consistently fit together, one puts space and time together as a geometric entity M called **spacetime**, a 4-dimensional (or in general $(n+1)$-dimensional) manifold. Each point p in spacetime is called an **event**. A curve $\gamma : I \to M$ in M is a smoothly varying points on M, with the direction of $\gamma' \in T_\gamma M$ encodes how "fast" it moves through spacetime.

For example, if $T_pM = \mathbb{R}^4$ with an orthonormal (in the Euclidean sense) basis $\{e_0, e_1, e_2, e_3\}$ and suppose e_0 is the time direction for some observer, then a vector $v = v_0e_0 + \cdots + v_3e_3 \in \mathbb{R}^4$ has a classical velocity $\left(\frac{v_1}{v_0}, \frac{v_2}{v_0}, \frac{v_3}{v_0}\right)$.

In this example, the coefficients v_0 measures the time displacement and $v_{1,2,3}$ measures the spacial displacement of a vector v (a geometric object). The values of the coefficients depends on the basis $\{e_0, e_1, e_2, e_3\}$ (how fast the observer moves and how the observer is oriented, *i.e.* the inertial frames of reference). Although it is more intuitive (in the classical sense) to formulate laws of physics in terms of the values $v_{0,1,2,3}$, the first postulate of Einstein means that the laws of physics should depend only on the geometric object v rather than the coefficients $v_{0,1,2,3}$. Hence if we write everything "geometrically" on spacetime, we arrive at formulation that is relativistic.

Now, the second postulate of Einstein says that the measurement of speed of light is also geometric, *i.e.* independent of coordinate. How to come up with a framework so that it is true?

Before revealing the sequence of definitions, let us look at the example of $\mathbb{R}^4$. If a vector $v \in \mathbb{R}^4$ is tangent to a light trajectory, we must have $c^2 = \left(\frac{v_1}{v_0}\right)^2 + \left(\frac{v_2}{v_0}\right)^2 + \left(\frac{v_3}{v_0}\right)^2$. In other words,

$$-c^2(v_0)^2 + (v_1)^2 + (v_2)^2 + (v_3)^2 = 0.$$

If we define for each $u, v \in T_pM$ the "inner product"

$$\langle u, v\rangle_L := -c^2u_0v_0 + u_1v_1 + u_2v_2 + u_3v_3$$

then we may use $\langle v, v\rangle_L$ as a measuring device for testing whether a tangent direction v is a light direction. We call v **light-like** if $\langle v, v\rangle_L = 0$. The collection of all light-like vectors in T_pM, as the solutions of the quadratic polynomial $\langle v, v\rangle_L = 0$, forms a cone called **light cone**. We call v **time-like** if $\langle v, v\rangle_L < 0$ (slower than speed of light, *i.e.* v lies in the solid interior of the light cone); and we call v **space-like** if $\langle v, v\rangle_L > 0$. See Figure 5.2.

It turns out $\langle ., .\rangle_L$ can be thought of as a "metric" for the spacetime manifold M. Using such a metric we may talk about geometries of embedded curves (which are particle path/history) analogous to space curves (curves embedded in $\mathbb{R}^3$); in terms of geometry of curves in spacetime the motion law will be made relativistic. With the notion of metric, we may also talk about the Hodge star $*$ for differential forms on M which will bring us relativistic Maxwell's equations.

Definition 5.1 (Lorentzian manifold)**.** Let M be an $(n+1)$-dimensional smooth manifold. It is **Lorentzian** if it is equipped with $\langle ., .\rangle_L$ defined (smoothly) on every tangent space

$$\langle ., .\rangle_L\big|_p : T_pM \times T_pM \to \mathbb{R}$$

such that

- $\langle ., .\rangle_L$ is bilinear and symmetric ($\langle u, v\rangle_L = \langle v, u\rangle_L$).

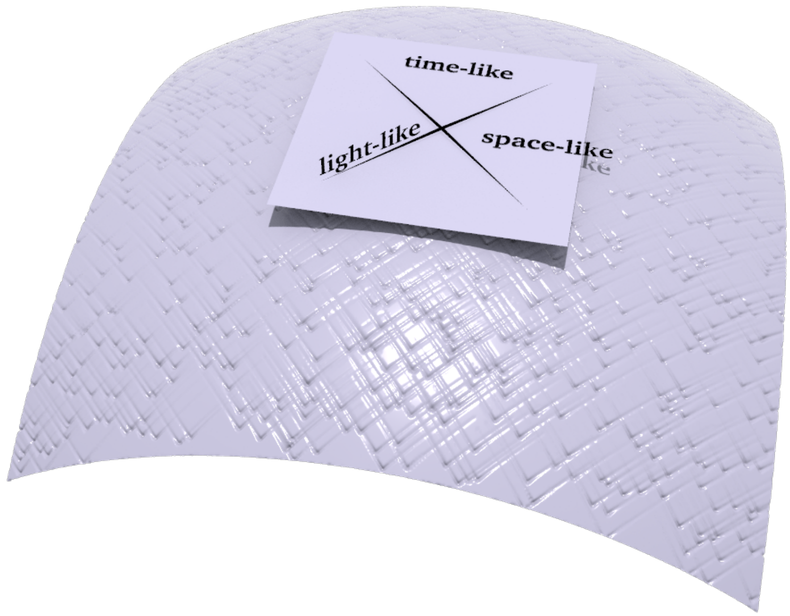


Figure 5.2: *A spacetime manifold of dimension* $1+1$. *Each tangent space has a distinct* ***light cone*** *(collection of light-like vectors) that separates time-like vectors and space-like vectors. The texture of the surface shows a few trajectories of light in this spacetime.*

- $\langle .,. \rangle_L$ is non-degenerate; *i.e.* there is no $u \in T_pM$ such that $\langle u, v \rangle_L = 0$ for all $v \in T_pM$.
- $\langle .,. \rangle_L$ has **signature** $(1, n)$. That is for any basis $\{e_0, \ldots, e_n\}$ for T_pM, the matrix $[\langle e_i, e_j \rangle_L]_{ij}$ has exactly one negative eigenvalue and n positive eigenvalues.

$\langle .,. \rangle_L$ is called a **Lorentzian metric** or **Lorentzian inner product**.

Exercise 5.11. Check that the signature of the above definition is independent of the choice of basis.

Definition 5.2. Let M be an $(n+1)$-dimensional Lorentzian manifold. Then $u \in T_pM$ is said to be

- **time-like** if $\langle u, u \rangle_L < 0$;
- **light-like** if $\langle u, u \rangle_L = 0$;
- **space-like** if $\langle u, u \rangle_L > 0$.

A vector $u \in T_pM$ is a **unit vector** if $\langle u, u \rangle_L = -c^2$ or $\langle u, u \rangle_L = 1$. A curve $\gamma \colon I \to M$ is said to be a **world line** if γ' is time-like (a path of moving object never exceeding speed of light). A world line $\gamma \colon \tau \mapsto \gamma(\tau)$ is **parameterized by arclength** if $\langle \gamma', \gamma' \rangle_L = -c^2$, in which case the parameter τ is called the **proper time**.

Similar to the Curve Chapter, we shall assume all world lines are parameterized by arclength (parameterized by proper time). An observer is a (arclength-parameterized) world line $\gamma : [0, \tau_{\max}] \to M$, and the proper time τ is his or her biological clock (or any other clock he or she brings along). Let $T = \gamma' \in T_\gamma M$, which is the time direction of the observer γ. Let $T^\perp = \{w \in T_\gamma M \mid \langle w, T\rangle_L = 0\}$ be the n-dimensional orthogonal complement. Note that $T^\perp$ consists of space-like vectors (can you see why?) which is the *space* experienced by the observer γ. Define for each $u \in T_\gamma M$

$$u_T = \frac{\langle T,u\rangle_L}{\langle T,T\rangle_L} = -\frac{1}{c^2}\langle T, u\rangle_L \quad (\textbf{Lorentz factor})$$
$$u_{T^\perp} = u - u_T T \in T^\perp.$$

Then every $u \in T_\gamma M$ can be orthogonally decomposed into

$$u = u_T T + u_{T^\perp}.$$

(The Lorentz factor is often denoted by "γ" in special relativity. Since the letter "γ" coincides with the notation for curves, and that the Lorentz factor totally depends on both u and the observer's direction T, and that it is the projection factor from u to T, we shall use the notation "u_T".)

Definition 5.3. Let γ be an observer, $T = \gamma'$, and suppose $u \in T_\gamma M$. Then the **classical velocity** of u observed by γ is given by

$$\frac{u_{T^\perp}}{u_T} \in T^\perp,$$

which is the space displacement divided by the time displacement.

Now we can check that Einstein's 2nd postulate is satisfied. Suppose u is a light-like vector ($\langle u, u\rangle_L = 0$). For any observer γ with $T = \gamma'$, one has

$$u = u_T T + u_{T^\perp}.$$

From $\langle u, u\rangle_L = 0$, using $\langle T, u_{T^\perp}\rangle_L = 0$ we get

$$0 = u_T^2 \langle T, T\rangle_L + \langle u_{T^\perp}, u_{T^\perp}\rangle_L = -c^2 u_T^2 + |u_{T^\perp}|^2$$

Hence the classical speed of u is $\frac{|u_{T^\perp}|}{|u_T|} = c$. Therefore, the value of the speed of light measured by any observer is the same constant c.

Note Classically velocity or speed has the unit of m/s. The constancy of c allows us to think of the unit s be just another unit of length through the canonical identification

$$\begin{aligned} c: \mathbb{R}\,\mathrm{s} &\xrightarrow{\cong} \mathbb{R}\,\mathrm{m} \quad \text{linear}, \\ c \cdot 1\,\mathrm{s} &:= 299792458\,\mathrm{m} \quad \text{in our universe}. \end{aligned}$$

Many authors would choose the time unit as $\tilde{\mathrm{s}} = \frac{1}{299792458}\,\mathrm{s}$ so that $c = 1$. However we would like to keep c in our formulas to give the right sense of scales. Hence, let us agree on the following convention: "second" is a unit of length with $1\,\mathrm{s} = 299792458\,\mathrm{m}$, and c is the number $c = 299792458 \in \mathbb{R}$. In particular *velocities are dimensionless*.

5.5.3 Momentum and Force

Suppose a world line $\gamma: I \to M$ is a particle trajectory with tangent $u = \gamma'$ (assuming arclength parameterization $\langle u, u\rangle_L = -c^2$). The vector $u \in T_{\gamma(\tau)}M$ is also called the **4-velocity**.

Let $m_0 \in \mathbb{R}\,\mathrm{kg}$, $m_0 > 0$, be the **rest mass** of the particle. Then the **4-momentum** is given by

$$P := m_0 u.$$

(The physical unit of P is $[P] = [\mathrm{kg}]$) And the **force** is given by

$$f := P' = m_0 u'.$$

($[f] = [\mathrm{kg}/\mathrm{m}]$) As a consequence of $\langle u, u\rangle_L = -c^2$ being constant, we always have $\langle f, u\rangle_L = 0$. In fact u', the **acceleration**, is exactly the curvature normal for the curve γ. The meaning of u' is clear when $M = \mathbb{R}^4$ analogous to space curves. When M is curved, the curvature of γ is referred as the **geodesic curvature** of the curve on M.

How does momentum and force "looks like" classically? Suppose there is an observer who travels at a direction T, $\langle T, T\rangle_L = -c^2$. Then

$$\begin{aligned} P &= m_0 u_T T + m_0 u_{T^\perp} \\ &= m_T T + m_T \tfrac{u_{T^\perp}}{u_T} \end{aligned}$$

where $m_T := m_0 u_T$ is the **classical mass** and $m_T \frac{u_{T^\perp}}{u_T} = P_{T^\perp}$ is the **classical momentum** (classical mass times classical velocity). Similarly,

$$f = m_T' T + f_{T^\perp}$$

assuming $T' = 0$. Here $f_{T^\perp} = P'_{T^\perp}$, so the **classical force** is given by $f_c := \frac{f_{T^\perp}}{u_T}$ (change of classical momentum per unit time displacement). Now, using $\langle f, u\rangle_L = 0$, one has

$$0 = m_T' \langle T, u\rangle_L + u_T \langle f_c, u\rangle_L$$

$$= -m_T' c^2 u_T + u_T \langle f_c, u_T T + u_{T\perp} \rangle_L$$
$$= -m_T' c^2 u_T + u_T^2 \left\langle f_c, \frac{u_{T\perp}}{u_T} \right\rangle.$$

This implies the **classical work rate** $\left\langle f_c, \frac{u_{T\perp}}{u_T} \right\rangle$ equals to $\frac{m_T'}{u_T} c^2$ where $\frac{m_T'}{u_T}$ classical rate of mass gain. Since work is the energy added into the particle, it associates an energy E_T to a mass m_T via

$$E_T = m_T c^2.$$

Thus, in the observer's expansion of the momentum $P = m_T T + m_T \frac{u_{T\perp}}{u_T}$, while the space component $m_T \frac{u_{T\perp}}{u_T}$ is understood as the classical momentum, the time component $m_T = \frac{E_T}{c^2}$ is understood as the energy.

The value observed energy $E_T = m_T c^2$ can be written in terms of m_0 and the information $u_T, u_{T\perp}$ of their relative speed. From $\langle P, P \rangle_L = -m_0^2 c^2$, one has

$$-c^2 m_T^2 + \left| m_T \frac{u_{T\perp}}{u_T} \right|^2 = -m_0^2 c^2$$

and thus obtains

$$E_T^2 = m_0^2 c^4 + c^2 \left| m_T \frac{u_{T\perp}}{u_T} \right|^2$$

i.e. the observed energy is contributed from the rest mass and the observed kinetic energy.

5.5.4 Maxwell's Equations

On the spacetime manifold M, suppose γ is a world line representing the trajectory of a charged particle with charge q. Let $u = \gamma'$ (again, $\langle u, u \rangle_L = -c^2$). Assume that the particle feels no other forces than the electromagnetic force. Then the force $f = m_0 u'$ is proportional to q and it is orthogonal to u. The latter statement can be rephrased by that $\iota_u f^\flat = 0$. Thus, $f^\flat$ *must be* the result of ι_u applied on some 2-form. (Can you see the image of ι_u equals to the kernel of ι_u?) Therefore, we conclude

$$f^\flat = -q \iota_u F$$

where F is a 2-form called the **Faraday 2-form**. ($F \in \Omega^2(M; \mathbb{R}^{\mathrm{kg/c}})$). If T is a unit time-like vector as the direction of an observer, then

$$E = \frac{1}{c^2} \iota_T F, \quad B = \jmath_{T\perp}^* F$$

where $\jmath_{T^\perp}^*$ is the pull-back for the inclusion $\jmath_{T^\perp}\colon T^\perp \hookrightarrow T_pM$. That is, B is the projection of F on $T^\perp$. See Example 5.3 in Appendix 5.A for more detail. Note that in fact the observer T decomposes F into

$$F = -T^\flat \wedge E + B.$$

Now we have F concisely represents both electric field and magnetic field. In terms of F, Maxwell's Equations are given by

$$\begin{cases} dF = 0 \\ d{*}F = J \end{cases}$$

where $J \in \Omega^n(M; \mathbb{R}\mathrm{C})$ (M is $(n+1)$-dimensional) necessarily satisfying $dJ = 0$ with

$$\rho = \tfrac{1}{c^2}\iota_T J, \quad j = \jmath_{T^\perp}^* F$$

being the classical charge density and charge current observed by T.

Appendix

5.A Contraction Operator

Given a tangent vector field v on M, the operator $\iota_v\colon \Omega^k(M) \to \Omega^{k-1}(M)$, called the **interior product** or **contraction with v**, is defined by

$$(\iota_v \omega)(X_1, \ldots, X_{k-1}) := \omega(v, X_1, \ldots, X_{k-1})$$

for $\omega \in \Omega^k(M)$ and $X_1, \ldots, X_{k-1}$ tangent vectors. In other words, ι_v is “inserting v into the first slot of a differential form”. The contraction product has the following properties similar to exterior derivative.

- $\iota_v \circ \iota_v = 0$.
- $\iota_v(\alpha \wedge \beta) = (\iota_v \alpha) \wedge \beta + (-1)^k \alpha \wedge (\iota_v \beta)$ when α is a k-form. (Leibniz rule)

In fact ι_v is the unique linear operator on differential forms so that $\iota_v^2 = 0$, Leibniz rule holds, and $\iota_v \alpha = \alpha(v)$ for all 1-form α.

Example 5.1. Let us consider $dx \wedge dy$ as a 2-form in $\mathbb{R}^3$. Then for a vector $v \in \mathbb{R}^3$, using Leibniz rule

$$\begin{aligned} \iota_v(dx \wedge dy) &= dx(v)\,dy - dy(v)\,dx \\ &= v_x\,dy - v_y\,dx. \end{aligned}$$

Example 5.2. Suppose B is a 2-form in $\mathbb{R}^3$ given by

$$B = B_x\,dy \wedge dz + B_y\,dz \wedge dx + B_z\,dx \wedge dy.$$

Let $v = (v_1, v_2, v_3)$ be a vector field. Then

$$(\iota_v B)^\sharp = -(v_x, v_y, v_z) \times (B_x, B_y, B_z).$$

A remarkable fact about the contraction is that it is the dual to a wedge product:

$$(\iota_v \alpha) \wedge *\beta = \alpha \wedge *(v^\flat \wedge \beta).$$

So, if you use the analogy $d \longleftrightarrow \iota_v$, then δ corresponds to $v^\flat \wedge$. Now you may think of what does Laplacian $\Delta = -d\,\delta - \delta\, d$ corresponds to in the ι_v version? It turns out that

$$(v^\flat \wedge) \circ \iota_v + \iota_v \circ (v^\flat \wedge) = |v|^2 \mathrm{id}.$$

If $|v|^2 = 1$, then the operators $\iota_v \circ (v^\flat \wedge)$ and $(v^\flat \wedge) \circ \iota_v$ are projection operators. One sees that in the evaluation $(\iota_v(v^\flat \wedge \omega))(X_1, \ldots, X_k)$, any v component of X_j will not contribute. If $\jmath\colon v^\perp \hookrightarrow T_p M$ is the inclusion map, then one would have that

$$\iota_v(v^\flat \wedge \omega)(X_1, \ldots, X_k) = (\jmath^* \omega)(X_1, \ldots, X_k)$$

for all $X_1, \ldots, X_k \in v^\perp$.

Example 5.3. In Section 5.5.4, if T is a unit time-like vector field ($\langle T, T \rangle_L = -c^2$), and F is the Faraday 2-form. Then

$$\begin{aligned} F &= -\tfrac{1}{c^2}\Big((T^\flat \wedge) \circ \iota_T + \iota_T \circ (T^\flat \wedge)\Big) F \\ &= -T^\flat \wedge \underbrace{\tfrac{1}{c^2} \iota_T F}_{E} + \underbrace{\jmath^*_{T^\perp} F}_{B} \end{aligned}$$

where E and B are the electric and magnetic fields measured by an observer traveling along a world line tangent to T.