Chapter 7

Hodge Decomposition

In the chapter on exterior calculus, we have seen that the property $d \circ d = 0$ unifies identities such as "$\operatorname{curl} \circ \operatorname{grad} = 0$", "$\operatorname{div} \circ \operatorname{curl} = 0$" of vector calculus. In this chapter we further let this property lead us to a useful tool – **Hodge decomposition**. A special case of Hodge decomposition is the **Helmholtz decomposition** in vector calculus widely applied in physics and engineering: any $\mathbb{R}^3$ vector field V on a simply-connected domain can be decomposed into

$$V = \operatorname{grad} u + \operatorname{curl} A,$$

where u and A are usually known as the scalar and vector potentials. Hodge decomposition is the exterior calculus version of the Helmholtz decomposition, which can apply to differential forms of any order on a manifold M of arbitrary dimensions. At a glance Hodge decomposition states: any k-form $\omega \in \Omega^k M$ can be decomposed into

$$\omega = d\alpha + \delta\beta + h$$

for some $\alpha \in \Omega^{k-1}M$, $\beta \in \Omega^{k+1}M$, and a **harmonic form** $h \in \Omega^k M$ satisfying $dh = 0$ and $\delta h = 0$. This remainder h shows up when M has nontrivial topology. For example on a domain with "holes", there are vector fields which cannot be written as the derivatives of scalar potentials and vector potentials. The richness of harmonic forms on a domain M, studied by the **de Rham cohomology theory**, is in fact related to "number of holes on M", which is algebraically measured by the **homology groups** of M. There are many equivalent versions of homology theory, and one of them, the **Morse homology**, has a discrete version which turned into an efficient algorithm known as the **tree-cotree algorithm** for finding the homology group of a discrete mesh.

We will develop the theory on closed manifolds, *i.e.*, compact manifolds without boundary, and later extend it to manifolds with boundaries.

7.1 Hodge Decomposition on Closed Manifolds

Suppose M is a closed manifold. Viewing each space of differential forms $\Omega^k M$ as a linear space, the exterior derivative d forms the following sequence of linear maps

$$\cdots \xrightarrow{d_{k-2}} \Omega^{k-1}M \xrightarrow{d_{k-1}} \Omega^k M \xrightarrow{d_k} \Omega^{k+1}M \xrightarrow{d_{k+1}} \cdots \tag{7.1}$$

Such a sequence of linear maps (or in general homomorphisms) from one vector space (abelian group or module) to the next is called a **chain complex** if $d_k \circ d_{k-1} = 0$, which is the case here. Specifically, the chain complex (7.1) of differential forms and exterior derivatives is called the **de Rham complex**.

Likewise,

$$\cdots \xleftarrow{\delta_{k-1}} \Omega^{k-1}M \xleftarrow{\delta_k} \Omega^k M \xleftarrow{\delta_{k+1}} \Omega^{k+1}M \xleftarrow{\delta_{k+2}} \cdots$$

is a chain complex.

Definition 7.1 ((Co)closed and (co)exact)**.** A k-form ω is said to be

- **closed** if $d\omega = 0$.
- **exact** if $\omega = d\alpha$ for some $(k-1)$-form α.
- **coclosed** if $\delta\omega = 0$.
- **coexact** if $\omega = \delta\beta$ for some $(k+1)$-form β.
- a **harmonic field** if ω is closed and coclosed.

Denote $\mathcal{H}^k M := \ker(d_k) \cap \ker(\delta_k) \subset \Omega^k M$ the set of all k-harmonic fields.

For instance, $\operatorname{im} d_{k-1}$ is the space of exact k-forms, $\ker d_k$ is the space of closed k-forms.

One thing we can infer from the chain complex is that $\operatorname{im} d_{k-1} \subset \ker d_k$; that is, *exact forms are closed*. Also note that d_k and δ_{k+1} are adjoint operators with respect to the L^2 inner product $\langle\!\langle .,. \rangle\!\rangle$, which tells us $\ker d_k = (\operatorname{im} \delta_{k+1})^\perp$.

Exercise 7.1 (Four fundamental subspaces)**.** Suppose V and W are linear spaces with inner product $\langle .,. \rangle_V$ and $\langle .,. \rangle_W$ respectively. Let $A : V \to W$ be a linear map and A^* be its adjoint. Check $(\operatorname{im} A^*)^\perp = \ker A$ and $(\operatorname{im} A)^\perp = \ker A^*$.

Therefore, by decomposition of orthogonal complements

$$\Omega^k M = \ker d_k \oplus \operatorname{im} \delta_{k+1}.$$

Since $\operatorname{im} d_{k-1} \subset \ker d_k$ is a subspace, we can further decompose orthogonal complements within $\ker d_k$

$$\ker d_k = \operatorname{im} d_{k-1} \oplus \left((\operatorname{im} d_{k-1})^\perp \cap \ker d_k\right)$$

$$
\begin{aligned}
&= \operatorname{im} d_{k-1} \oplus (\ker \delta_k \cap \ker d_k) \\
&= \operatorname{im} d_{k-1} \oplus \mathcal{H}^k M.
\end{aligned}
$$

Putting the above together we have

Theorem 7.1 (Hodge decomposition on closed manifolds)**.**

$$\Omega^k M = \operatorname{im} d_{k-1} \oplus \operatorname{im} \delta_{k+1} \oplus \mathcal{H}^k(M).$$

That is, for each $\omega \in \Omega^k M$, there is a unique $\alpha \in \Omega^{k-1} M$ up to a closed form, a unique $\beta \in \Omega^{k+1} M$ up to a coclosed form, and a unique $h \in \mathcal{H}^k M$, such that

$$\omega = d\alpha + \delta\beta + h.$$

The harmonic field part $\mathcal{H}^k M$ in the Hodge decomposition can be understood by the following lemma.

Lemma 7.1 (Harmonic forms on closed manifolds)**.** *On a closed manifold M,*

$$\mathcal{H}^k(M) = \left\{ h \in \Omega^k M \;\middle|\; \Delta h = 0 \right\}, \quad \textit{where } \Delta = -d\,\delta - \delta\, d.$$

That is, harmonic fields are the solutions to the Laplace's equation, and vice versa.

Exercise 7.2. Prove Lemma 7.1.

Exercise 7.3. Give a counterexample for Lemma 7.1 when M has a boundary.

7.1.1 Solving for the Exact, Coexact and Harmonic Components

Theorem 7.1 says that for ω a k-form on a closed manifold M,

$$\omega = d\alpha + \delta\beta + h. \tag{7.2}$$

So how do we find $d\alpha$, $\delta\beta$ and h in practice? The following examples take us through the standard procedure for extracting each component of Hodge decomposition.

Example 7.1 (Exact component)**.** To find the exact component $d\alpha$ in (7.2), apply δ on both sides of (7.2) and get

$$\delta\, d\alpha = \delta\omega. \tag{7.3}$$

To understand the solvability of (7.3), let us analyze this linear equation by investigating the image, kernel of the linear operator $\delta\, d$ and its adjoint, *i.e.* the four fundamental subspaces of $\delta\, d$.

Fundamental Subspaces of $\delta\, d$ First, note that $\delta\, d$ (more precisely $\delta_k\, d_{k-1}$) is self-adjoint:

$$\langle\!\langle \delta\, d u, v\rangle\!\rangle = \langle\!\langle d u, d v\rangle\!\rangle = \langle\!\langle u, \delta\, d v\rangle\!\rangle \quad \forall u, v \in \Omega^{k-1}(M).$$

Therefore, from the orthogonal relations among the four fundamental subspaces (Exercise 7.1), the image and kernel of $\delta_k\, d_{k-1}$ are orthogonal complements in $\Omega^{k-1}(M)$:

$$\ker(\delta_k\, d_{k-1})^{\perp} = \operatorname{im}(\delta_k\, d_{k-1}).$$

Next, we claim that in fact the image and kernel are parts of the Hodge decomposition for $\Omega^{k-1}(M)$:

$$\begin{cases} \operatorname{im}(\delta_k\, d_{k-1}) = \operatorname{im}(\delta_k) \\ \ker(\delta_k\, d_{k-1}) = \operatorname{im}(d_{k-2}) \oplus \mathcal{H}^{k-1}(M) = \ker(d_k). \end{cases} \tag{7.4}$$

To prove this claim, we start with computing the kernel. Suppose $u \in \ker(\delta_k d_{k-1})$, *i.e.* $\delta\, d u = 0$. Write u in terms of its Hodge decomposition

$$u = d\,\phi + \delta\psi + \chi$$

where $\chi \in \mathcal{H}^{k-1}(M)$. Then we find that $0 = \delta\, d u = \delta\, d\, \delta\psi$, which implies that

$$0 = \langle\!\langle \delta\, d\, \delta\psi, \delta\psi\rangle\!\rangle = \langle\!\langle d\, \delta\psi, d\, \delta\psi\rangle\!\rangle = \|\, d\, \delta\psi\|^2.$$

Hence $d\, \delta\psi = 0$. This further implies $0 = \langle\!\langle d\, \delta\psi, \psi\rangle\!\rangle = \langle\!\langle \delta\psi, \delta\psi\rangle\!\rangle$. Therefore $\delta\psi = 0$, and thus $u \in \operatorname{im}(d_{k-2}) \oplus \mathcal{H}^{k-1}(M)$. Conversely, any element in $\operatorname{im}(d_{k-2}) \oplus \mathcal{H}^{k-1}(M)$ is clearly in $\ker(\delta\, d)$. Therefore, we have shown that $\ker(\delta_k\, d_{k-1}) = \operatorname{im}(d_{k-2}) \oplus \mathcal{H}^{k-1}(M)$.

From $\operatorname{im}(\delta\, d) = \ker(\delta\, d)^{\perp}$ we obtain $\operatorname{im}(\delta\, d) = \operatorname{im}(\delta)$ using once again the Hodge decomposition of u above.

Back to Eq. (7.3) The analysis above tells us that the RHS of (7.3) is in the image of $\delta\, d$. Therefore a solution α must exist. We also know from (7.4) that α is unique only up to $\ker(\delta\, d) = \ker(d)$. (Remember in the end we are interested in $d\,\alpha$, and we will really obtain a unique $d\,\alpha$!) Although any of the infinitely many solutions α for (7.3) is equally valid, in practice, one often finds *the* α that has the least norm (*i.e.* the minimal solution).

Coulomb Gauge The "canonicalized" choice for minimal solution α to (7.3) is called **Coulomb gauge**. Note that α itself has the Hodge (orthogonal) decomposition

$$\alpha = \underbrace{\chi + d\phi}_{\in\ker(\delta d)} + \underbrace{\delta\psi}_{\in\operatorname{im}(\delta d)}.$$

Therefore α under Coulomb gauge has neither harmonic nor exact component (in order to minimize $\|\alpha\|^2 = \|\chi\|^2 + \|d\phi\|^2 + \|\delta\psi\|^2$). Adding these conditions to (7.3) nails down a unique solution. This *well-posed* version of (7.3) is given by

$$\begin{cases} \delta d\alpha = \delta\omega \\ \delta\alpha = 0 \\ \langle\!\langle \alpha, \chi \rangle\!\rangle = 0 \quad \text{for all } \chi \in \mathcal{H}^{k-1}(M). \end{cases} \tag{7.5}$$

An equivalent way of writing (7.5) takes a slightly more concise form (a Poisson problem!)

$$\begin{cases} -\Delta\alpha = \delta\omega \\ \langle\!\langle \alpha, \chi \rangle\!\rangle = 0 \quad \text{for all } \chi \in \mathcal{H}^{k-1}(M) \end{cases} \tag{7.6}$$

where $\Delta = -\delta d - d\delta$. Now, one has to show that (7.5) is equivalent to (7.6). It is obvious that the solution to (7.5) satisfies (7.6). Conversely, suppose α is a solution to (7.6). Then

$$\delta d\alpha + d\delta\alpha = \delta\omega.$$

Since the RHS is in $\operatorname{im}(\delta)$, by Hodge decomposition we conclude that $d\delta\alpha = 0$. This implies $0 = \langle\!\langle d\delta\alpha, \alpha \rangle\!\rangle = \|\delta\alpha\|^2$ and $\delta d\alpha = \delta\omega$. Hence α satisfies (7.5).

Example 7.2 (Coexact component). To find β in (7.2), we solve for a solution to

$$d\delta\beta = d\omega. \tag{7.7}$$

One can check in a way similar to the procedure in Example 7.1 that Eq. (7.7) has a solution which is unique up to $\ker\delta$. Among all solutions we aim for the solution β with least norm, *i.e.* the solution under Coulomb gauge. This solution under Coulomb gauge is also the solution to the problem

$$\begin{cases} d\delta\beta = d\omega \\ d\beta = 0 \\ \langle\!\langle \beta, \chi \rangle\!\rangle \quad \text{for all } \chi \in \mathcal{H}^{k+1}(M), \end{cases} \tag{7.8}$$

or the equivalent Poisson problem

$$\begin{cases} -\Delta\beta = d\,\omega \\ \langle\!\langle \beta, \chi \rangle\!\rangle = 0 \quad \text{for all } \chi \in \mathcal{H}^{k+1}(M). \end{cases} \tag{7.9}$$

The problems (7.8) and (7.9) are well posed. Each of them admits a unique solution.

Example 7.3 (Find the harmonic component)**.** After finding $d\,\alpha$ and $\delta\beta$ components of $\omega = d\,\alpha + \delta\beta + h$, simply use

$$h = \omega - d\,\alpha - \delta\beta$$

to extract the harmonic component.

Alternatively, one may first find a basis for $\mathcal{H}^k(M)$ (it is a finite dimensional space, and a method for finding its basis will be discussed in the end of this chapter). And then using this basis, orthogonally project ω onto $\mathcal{H}^k(M)$ to extract the harmonic part.

Algorithm 1 Hodge decomposition

Input: $\omega \in \Omega^k(M)$ ▷ $\partial M = \emptyset$
  $\alpha \leftarrow$ Solve either $\delta\,d\,\alpha = \delta\omega$ or $-\Delta\alpha = \delta\omega$.
  $\beta \leftarrow$ Solve either $d\,\delta\beta = d\,\omega$ or $-\Delta\beta = d\,\omega$.
  $h \leftarrow \omega - d\,\alpha - \delta\beta$.
  return $d\,\alpha$, $\delta\beta$, h.

Hodge decomposition is a powerful tool for asserting well-posedness of certain PDE problems. For example, given an force field $f \in \Omega^1(M;\mathbb{R}\mathrm{J})$ can we find a scalar potential $u \in \Omega^0(M;\mathbb{R}\mathrm{J})$ such that $-du = f$? (Is f a conservative force?) If there exists such u, then how many?

Example 7.4 (Well-posedness of $d\,\alpha = \omega$)**.** On a closed manifold M,

$$d\,\alpha = \omega$$

has a solution for α if and only if $\omega \in \Omega^k(M)$ satisfies the conditions

$$\begin{cases} d\,\omega = 0 \\ \langle\!\langle \omega, h \rangle\!\rangle = 0 \quad \text{for all } h \in \mathcal{H}^k(M). \end{cases}$$

The solution is unique up to a closed $(k-1)$-form. That is, if α_0 is a solution to $d\,\alpha = \omega$, then α is also a solution if and only if $\alpha = \alpha_0 + d\,\phi + \chi$ for $\phi \in \Omega^{k-2}(M)$ and $\chi \in \mathcal{H}^{k-1}(M)$.

In the case of finding a scalar potential, we are interested in the non-uniqueness of scalar potentials, which is given by $\mathcal{H}^0(M)$ consisting of 0-forms with vanishing d.

Example 7.5. If M is closed and path connected, then $\mathcal{H}^0(M) = \mathbb{R}$.

Example 7.6. Suppose M is closed. Then $\mathcal{H}^0(M) = \mathbb{R}^c$ where c is the number of connected component of M.

When studying potential forces in classical physics, we often say "potential energy is unique up to a constant (per connected component)". Now we see this is a direct consequence of Hodge and de Rham theory. You may have noticed that the dimension of the space $\mathcal{H}^0(M)$ depends only on the topology of M (number of connected components).

7.2 De Rham Cohomology

The harmonic forms appearing in Hodge decomposition are the closed forms that are not exact (or equivalently coclosed forms that are not coexact). The space of such forms is abstractly understood as the **(co)homology class**, and it is determined from the topology of the domain M. In this section we introduce **de Rham cohomology** given as the class of closed form that are not exact (*i.e.*, the space of harmonic forms). Remember, in the discrete setting, the exterior derivative operator d is the transpose of the boundary operator (adjacency matrix) ∂. This particular relation between derivative and boundary operators induces a relation between the de Rham cohomology and its analogy for embedded geometries – **simplicial homology**, the class of closed (boundary-less) chains that are not the boundary of other chains. Later in this section we will see that simplicial homology is *isomorphic* to de Rham cohomology and therefore the space of harmonic forms. The task of seeking for a basis for the space of harmonic forms can be reformulated as finding for a basis (**generators**) for the simplicial homology.

Let us begin with the following chain complex

$$\cdots \xrightarrow{d_{k-2}} \Omega^{k-1}M \xrightarrow{d_{k-1}} \Omega^k M \xrightarrow{d_k} \Omega^{k+1}M \xrightarrow{d_{k+1}} \cdots .$$

Note that we always have $\operatorname{im} d_{k-1} \subset \ker d_k$. This means that exactness implies closedness. However, there might be closed forms that are not exact. We capture this "gap" from $\operatorname{im} d_{k-1}$ to $\ker d_k$ by the quotient space $(\ker d_k)/(\operatorname{im} d_{k-1})$.

Definition 7.2 (De Rham cohomology)**.** The **k-th de Rham cohomology** of a manifold M is the quotient space

$$H^k(M, d) := \frac{\ker d_k}{\operatorname{im} d_{k-1}}.$$

Likewise we denote

$$H_k(M,\delta) := \frac{\ker \delta_k}{\operatorname{im} \delta_{k+1}}$$

the k-th homology from the dual δ-chain complex.

Recall that the harmonic part $\mathcal{H}^k(M)$ in the Hodge decomposition on closed manifolds is separated from $\ker d_k$ by considering the orthogonal complement of $\operatorname{im} d_{k-1} \subset \ker d_k$. This means that $\mathcal{H}^k(M) \cong \frac{\ker d_k}{\operatorname{im} d_{k-1}}$.

Corollary 7.1 (Hodge isomorphism). *On a closed manifold M,*

$$\mathcal{H}^k(M) \cong H^k(M,d).$$

7.2.1 Simplicial Homology and Cohomology

The construction of the de Rham cohomology begins with a chain complex (sequence of maps with the property $\operatorname{im}(d_{k-1}) \subset \ker(d_k)$). Note that such a structure also holds in the discrete setting for chains (with ∂ as the sequence of maps) and cochains (with $\partial^\intercal$ as the sequence of maps). In analogy to the de Rham cohomology, the class of chains in $\ker(\partial)$ that are not in $\operatorname{im}(\partial)$ is called the **simplicial homology**, and the class of cochains in $\ker(\partial^\intercal)$ that are not in $\operatorname{im}(\partial^\intercal)$ is called the **simplicial cohomology**. As will be shown later, simplicial homology and simplicial cohomology are isomorphic. It is also true that the simplicial cohomology is just the discrete version of de Rham cohomology. Therefore, the study of the de Rham cohomology for differential forms can be done through understanding its counterpart for chains (which are just points, curves, surfaces *etc.*).

Recall that a formal linear combinations of k-simplices is called a **k-(simplicial) chain** denoted by $C_k(M;\mathbb{R})$. The dual space of $C_k(M;\mathbb{R})$ is denoted by $C^k(M;\mathbb{R})$, whose element is called a **k-(simplicial) cochain** or a **discrete k-form**, *i.e.* values sitting on k-simplices. Elements in C_k or C^k can be represented by a vector of numbers. The **boundary operator** $\partial_k\colon C_k(M;\mathbb{R}) \to C_{k-1}(M;\mathbb{R})$ is represented as an n_{k-1}-by-n_k (sparse) adjacency matrix with appropriate signs. The transpose $\partial_k^\intercal\colon C^{k-1}(M;\mathbb{R}) \to C^k(M;\mathbb{R})$ is the **coboundary operator** or the **discrete exterior derivative**.

Simplicial homology

On simplicial chains, the boundary operator ∂ are the sequence of maps

$$\cdots \xrightarrow{\partial_{k+2}} C_{k+1}(M;\mathbb{R}) \xrightarrow{\partial_{k+1}} C_k(M;\mathbb{R}) \xrightarrow{\partial_k} C_{k-1}(M;\mathbb{R}) \xrightarrow{\partial_{k-1}} \cdots$$

forming so called the **simplicial chain complex**. (Note that $\partial_k \circ \partial_{k+1} = 0$).

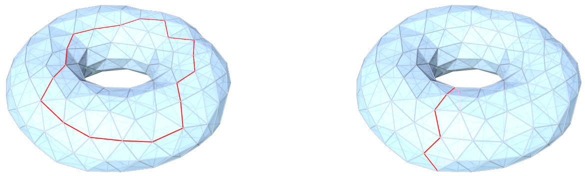

Figure 7.1: *Two examples of closed 1-chain that is not the boundary of any 2-chain.*

Definition 7.3 (Simplicial homology)**.** A simplicial chain Γ in $C_k(M;\mathbb{R})$ is

- **closed** if $\partial\Gamma = 0$, *i.e.* it has no boundary.
- **exact** if $\Gamma = \partial\Sigma$ for some $\Sigma \in C_{k+1}(M;\mathbb{R})$.

The **k-th simplicial homology** is then defined as the quotient space

$$H_k(M,\partial;\mathbb{R}) := \frac{\ker\partial_k}{\operatorname{im}\partial_{k+1}}.$$

An element in the k-th simplicial homology is an affine space

$$\Gamma + \operatorname{im}\partial_{k+1} = [\Gamma]$$

where Γ is a closed chain that is not the boundary of any $(k+1)$-chain.

Figure 7.1 shows two 1-chains Γ_1 and Γ_2 that are closed but not the boundaries of any 2-chain on a torus. Each of such 1-chain represents an element in $H_1(M,\partial;\mathbb{R})$. For example $[\Gamma_1] = \Gamma_1 + \operatorname{im}\partial_2 \in H_1(M,\partial;\mathbb{R})$ is the class of closed 1-chain that can be "deformed" to Γ_1 by adding a boundary of a 2-chain.

The 1-chain Γ_1 also *generates* a subspace in $H_1(M,\partial;\mathbb{R})$, namely the subspace consisting of $c[\Gamma_1] = c\Gamma_1 + \operatorname{im}\partial_2 \in H_1(M,\partial;\mathbb{R})$ for $c \in \mathbb{R}$.

Now here are two facts. First, $[\Gamma_2]$ is *linearly independent* to $[\Gamma_1]$. That is, $c_1[\Gamma_1] \neq c_2[\Gamma_2]$ for all $c_1, c_2 \in \mathbb{R}$ unless $c_1 = c_2 = 0$. The other fact is that, the linear space $H_1(M,\partial;\mathbb{R})$ is two dimensional consisting of

$$c_1[\Gamma_1] + c_2[\Gamma_2] = c_1\Gamma_1 + c_2\Gamma_2 + \operatorname{im}\partial_2, \quad c_1, c_2 \in \mathbb{R}.$$

Here $[\Gamma_1]$ and $[\Gamma_2]$ forms a basis for $H_1(M,\partial;\mathbb{R})$, in which case we call Γ_1, Γ_2 **homology generators**.

Definition 7.4 (Betti number)**.** The k-th **Betti number** is defined by

$$\beta_k := \operatorname{rank}(H_k(M,\partial;\mathbb{R})).$$

That is, $H_k(M,\partial;\mathbb{R}) \cong \mathbb{R}^{\beta_k}$.

In practice, a set of the 1st-homology generators for surfaces can be found by the **tree-cotree algorithm** explained in a later section.

Simplicial cohomology

On simplicial cochains, the **coboundary operator** $\partial^\intercal$, a.k.a. discrete exterior derivative d, yields

$$\cdots \xrightarrow[d_{k-2}]{\partial_{k-1}^\intercal} C^{k-1}(M;\mathbb{R}) \xrightarrow[d_{k-1}]{\partial_k^\intercal} C^k(M;\mathbb{R}) \xrightarrow[d_k]{\partial_{k+1}^\intercal} C^{k+1}(M;\mathbb{R}) \xrightarrow[d_{k+1}]{\partial_{k+2}^\intercal} \cdots$$

the **simplicial cochain complex**.

Definition 7.5 (Simplicial cohomology)**.** The ***k*-th simplicial cohomology** is defined by

$$H^k(M,\partial^\intercal;\mathbb{R}) := \frac{\ker \partial_{k+1}^\intercal}{\operatorname{im} \partial_k^\intercal} = \frac{\ker d_k}{\operatorname{im} d_{k-1}}.$$

The following theorem says that the simplicial cohomology defined here is in fact the same as the simplicial homology.

Theorem 7.2 (Universal Coefficient Theorem)**.** *The simplicial homology and cohomology (of real coefficient) are the same*

$$H^k(M,\partial^\intercal;\mathbb{R}) \cong \mathbb{R}^{\beta_k} \cong H_k(M,\partial;\mathbb{R}).$$

Proof. This can be directly shown by exploiting the 4 fundamental subspaces of matrices.

First of all, $H_k(M,\partial;\mathbb{R}) = \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}}$, so

$$\ker \partial_k \cong \operatorname{im} \partial_{k+1} \oplus H_k(M,\partial;\mathbb{R}).$$

We also have $C_k(M;\mathbb{R}) \cong \ker \partial_k \oplus \operatorname{im} \partial_k^\intercal$. Thus,

$$C_k(M;\mathbb{R}) \cong \operatorname{im} \partial_{k+1} \oplus \operatorname{im} \partial_k^\intercal \oplus H_k(M,\partial;\mathbb{R}).$$

We also have $C_k(M;\mathbb{R}) \cong \operatorname{im} \partial_{k+1} \oplus \ker \partial_{k+1}^\intercal$. Hence,

$$\ker \partial_{k+1}^\intercal \cong \operatorname{im} \partial_k^\intercal \oplus H_k(M,\partial;\mathbb{R}).$$

Therefore,

$$H^k(M,\partial^\intercal;\mathbb{R}) := \frac{\ker \partial^\intercal_{k+1}}{\operatorname{im} \partial^\intercal_k} \cong H_k(M,\partial;\mathbb{R}).$$

□

Remark 7.1. Our chains and cochains use *real* coefficient and form standard linear spaces. The full version of Universal Coefficient Theorem takes a more complicated form that holds for chains with R (some ring) coefficient and cochains with G coefficient where G is some module over R:

$$H^k(M,\partial^\intercal;G) \cong \hom_R\left(H_k(M,\partial;R);G\right) \oplus \operatorname{ext}_R\left(H_{k-1},(M,\partial;R);G\right).$$

We will consider only the real coefficient case, so that everything is just a matter of finite dimensional linear algebra.

7.2.2 De Rham Theorem

What do we have so far? We started with the simplicial homology $H_\bullet(M,\partial;\mathbb{R})$, which is the class of closed chains that are not the boundary of some other chain. Then we defined the simplicial cohomology $H^\bullet(M,\partial^\intercal;\mathbb{R})$ in analog to the construction of homology. By Universal Coefficient Theorem, we discovered that $H^\bullet(M,\partial^\intercal;\mathbb{R}) \cong H_\bullet(M,\partial;\mathbb{R}) \cong \mathbb{R}^{\beta_\bullet}$.

Now notice that the simplicial cohomology $H^\bullet(M,\partial^\intercal;\mathbb{R})$ is nothing but the de Rham cohomology $H^\bullet(M,d;\mathbb{R}) \cong \mathcal{H}^\bullet(M;\mathbb{R})$ (the space of harmonic forms) in the discrete exterior calculus setup. De Rham Theorem uses the *interpolation and sampling* for discrete and smooth exterior calculus to argue that the simplicial cohomology is indeed the same as the smooth de Rham cohomology.

Theorem 7.3 (De Rham Theorem)**.** *The de Rham cohomology is isomorphic to the simplicial cohomology*

$$H^k(M,d;\mathbb{R}) \cong H^k(M,\partial^\intercal;\mathbb{R})$$

induced by the sampling map

$$\int : \Omega^k(M;\mathbb{R}) \to C^k(M;\mathbb{R}).$$

Proof. The sampling map $\int : \Omega^k(M;\mathbb{R}) \to C^k(M;\mathbb{R})$ takes a k-form ω, and returns a cochain $\int_{(\cdot)} \omega$. Note that by Stokes' Theorem we have $\int \circ d = \partial^\intercal \circ \int$. Since the de Rham cohomology $H^k(M,d;\mathbb{R}) \cong \mathcal{H}^k(M;\mathbb{R})$ the space of harmonic k-forms, we are left to show that

$\int$ induces a linear isomorphism between $\mathcal{H}^k(M;\mathbb{R})$ and $H^k(M,\partial^\intercal,\mathbb{R})$. Consider mapping each harmonic field $h \in \mathcal{H}^k(M;\mathbb{R})$ to

$$\left[\int_{(\cdot)} h\right] := \int_{(\cdot)} h + \operatorname{im}\partial_k^\intercal \in \frac{\ker \partial_{k+1}^\intercal}{\operatorname{im}\partial_k^\intercal} = H^k(M).$$

Here we have used the fact that $\int_{(\cdot)} h + \operatorname{im}\partial_k^\intercal$, as an affine space in $C^k(M;\mathbb{R})$, lies in $\ker \partial_{k+1}^\intercal$ (using Stokes and that h is closed). Now we have a linear map $h \mapsto \left[\int h\right]$ from $\mathcal{H}^k(M;\mathbb{R})$ to $H^k(M,\partial^\intercal;\mathbb{R})$. This map has an inverse map: given any $[c] \in H^k(M)$, take a representative $c \in [c]$ and use the Whitney interpolation to come up with a differential form $\hat{c}$, from which extract its harmonic part by Hodge decomposition. The proof follows after checking that the inverse map is well-defined (independent of the choice of representative $c \in [c]$). $\square$

7.2.3 Summary for de Rham cohomology

Up to now we have the remarkable topological result: the following spaces are all isomorphic to each other.

(i) Simplicial homology $H_k(M,\partial;\mathbb{R})$, the class of closed k-chains that are not the boundary of any $(k+1)$-chain.
(ii) Simplicial cohomology $H^k(M,\partial^\intercal;\mathbb{R})$, the class of closed k-cochains that are not the coboundary of any $(k-1)$-cochain.
(iii) De Rham cohomology $H^k(M,d;\mathbb{R})$, the class of closed k-differential forms that are not exact.
(iv) Space of harmonic k-forms $\mathcal{H}^k(M)$.

The isomorphism between (i) and (ii) is given by Universal Coefficient Theorem, a direct application of 4 fundamental subspaces of matrices. The isomorphism between (ii) and (iii) is given by de Rham Theorem using sampling and interpolation between discrete and smooth exterior calculus. The isomorphism between (iii) and (iv) is the result of Hodge decomposition.

Since all homology/cohomology we have introduced are the same, we may just abbreviate the homology as

$$H_k(M) \quad \text{or} \quad H^k(M).$$

The following example combines a few results here to give an algorithm for creating a basis for harmonic fields $\mathcal{H}^k(M)$ given a set of generators of $H_k(M)$.

Example 7.7 (Primal chain generators to harmonic basis)**.** Suppose $\Gamma_1, \ldots, \Gamma_{\beta_k}$ are k-chains being a set of generators for $H_k(M)$. Each Γ_i is a combinations of simplices σ_j with an appropriate sign depending on the orientation

$$\Gamma_i = \sum_{j=1}^{n_k} s_{ij}\sigma_j \longleftrightarrow \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{in_k} \end{bmatrix}$$

where $s_{ij} \in \{0, \pm 1\}$ is the indicator whether Γ_i includes simplex j. Now for each generator Γ_i, consider a discrete differential k-form (cochain) ω_i defined so that

$$\omega_i(\sigma_j) = s_{ij}.$$

The resulting k-form is coclosed. (Why?) Apply Hodge decomposition to ω_i to extract its harmonic component h_i. The resulting (discrete) harmonic fields $\{h_i\}_{i=1}^{\beta_k}$ form a basis for $\mathcal{H}^k(M)$.

7.3 Poincaré Dual

On an oriented n-manifold, the Hodge star $*$ allows us to link k-forms and $(n-k)$-forms.

Recall that the codifferential $\delta_k = (-1)^k *^{-1} d*$ yields the following chain complex

$$\cdots \longrightarrow \Omega^{k+1}M \xrightarrow{\delta_{k+1}} \Omega^k M \xrightarrow{\delta_k} \Omega^{k-1}M \longrightarrow \cdots$$

from which we have the homology for codifferential

$$H_k(M,\delta;\mathbb{R}) \coloneqq \frac{\ker \delta_k}{\operatorname{im} \delta_{k+1}}$$

the class of coclosed forms that are not coexact. This space is the same as the space of harmonic fields

$$H_k(M,\delta;\mathbb{R}) \cong \mathcal{H}^k(M;\mathbb{R})$$

using the same argument (Hodge decomposition) for showing $\mathcal{H}^k(M;\mathbb{R}) \cong H^k(M,d;\mathbb{R})$.

The following Theorem uses the Hodge star taking us from k-harmonic fields to $(n-k)$-harmonic fields.

Theorem 7.4 (Poincaré-Hodge duality)**.** *On a closed oriented n-dimensional manifold M, the Hodge star isomorphism $* : \Omega^k M \xrightarrow{\cong} \Omega^{n-k} M$ induces an isomorphism*

$$H^k(M, d; \mathbb{R}) \cong H_{n-k}(M, \delta; \mathbb{R}).$$

Proof. By $\delta = (-1)^k *^{-1} d *$, the chain complex

$$\Omega^{n-k+1} M \xrightarrow{\delta_{n-k+1}} \Omega^{n-k} M \xrightarrow{\delta_{n-k}} \Omega^{n-k-1} M$$

is just

$$\begin{tikzcd} \Omega^{k-1} M \arrow[r, "d_{k-1}"] & \Omega^k M \arrow[r, "d_k"] & \Omega^{k+1} M \\ \Omega^{n-k+1} M \arrow[u, "*^{-1}"', "\wr\|"] & \Omega^{n-k} M \arrow[u, "*^{-1}"', "\wr\|"] & \Omega^{n-k-1} M \arrow[u, "*^{-1}"', "\wr\|"] \end{tikzcd}$$

and hence identical to

$$\Omega^{k-1} M \xrightarrow{d_{k-1}} \Omega^k M \xrightarrow{d_k} \Omega^{k+1} M.$$

□

Corollary 7.2. *On a closed oriented n-dimensional manifold M with a simplicial structure,*

$$H_k(M) \cong H^k(M) \cong \mathcal{H}^k(M) \cong \mathcal{H}^{n-k}(M) \cong H^{n-k}(M) \cong H_{n-k}(M).$$

In other words, the Betti numbers satisfy $\beta_k = \beta_{n-k}$.

Example 7.8 (Generators in dual mesh to harmonic basis)**.** Suppose M is a simplicial mesh, and suppose M' is its dual mesh. Though M' is not simplicial, one still has the similar notion of homology and cohomology using its corresponding DEC. Suppose $\Gamma = \sigma_1^* + \cdots \sigma_m^*$ be a generator for $H_k(M')$. Consider an $(n-k)$-primal form ω defined so that

$$\omega(\sigma_j) = 1 \quad \text{for } \sigma_j^* \text{ in } \Gamma$$

and $\omega(\sigma) = 0$ for all other $(n-k)$-simplex σ. The resulting $(n-k)$-form ω is closed. (Why?) Apply Hodge decomposition to ω and extract its harmonic component h.

For linearly independent generators $\Gamma_1, \ldots, \Gamma_{\beta_k}$, the constructed harmonic $(n-k)$-forms $h_1, \ldots, h_k$ will be linearly independent and form a basis for $\mathcal{H}^{n-k}(M)$.

7.4 The Betti Numbers

The Betti number β_k, defined in Definition 7.4, is the dimension of the k-th (co)homology of M. On oriented n-manifold, $\beta_k = \beta_{n-k}$ from Poincaré duality.

Each Betti number encodes some topological information about the domain M:

- The zeroth Betti number β_0 is always the number of connected component of M. (Why?) So in most cases when M is connected, $\beta_0 = 1$.
- The first Betti number β_1 is the number of kinds of closed loops in M that are not the boundary of some surface.
- The second Betti number β_2 is the number of types of closed surfaces in M that are not the boundary of some volume.
- For $m > n$ we have $\beta_m = 0$. (Why?)

Here are two beautiful and useful facts about the Betti numbers which we will not prove.

Theorem 7.5 (Betti numbers and Euler Characteristic). *Suppose $\chi(M)$ is the Euler characteristic of M. Then*

$$\chi(M) = \sum_{k=0}^{\infty} (-1)^k \beta_k.$$

In many of computational applications M is an oriented closed surface. Hence the result of the following example is very important.

Example 7.9. A genus-g closed and oriented surface M has Euler characteristic $\chi(M) = 2 - 2g$. Since M is connected and oriented, we have $\beta_0 = \beta_2 = 1$. Hence

$$\beta_1 = 2g.$$

Another useful fact is about the Poicaré polynomial.

Definition 7.6 (Poincaré polynomial). The **Poincaré polynomial** of a domain M with Betti numbers $\{\beta_k\}_{k=0}^{\infty}$ is defined by the generating function

$$P(t) := \sum_{k=0}^{\infty} \beta_k t^k.$$

Theorem 7.6 (Künneth Theorem). *Let M, N be two domains with Poincaré polynomials P_M, P_N respectively. Then the product space $M \times N$ has Poincaré polynomial*

$$P_{M\times N}(t) = P_M(t) P_N(t).$$

Example 7.10. A torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ where $\mathbb{S}^1$ is a single circle. Since $\mathbb{S}^1$ is oriented, connected and 1-dimensional, $P_{\mathbb{S}^1}(t) = 1 + t$ (that is $\beta_0 = \beta_1 = 1$). So it follows that $P_{\mathbb{T}^2} = (1+t)^2 = 1 + 2t + t^2$. Hence the first Betti number of a torus is 2.

Example 7.11. The n-torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \mathbb{S}^1$ (n times) has Poicaré polynomial given by $(1+t)^n$. That is, $\beta_k = \binom{n}{k}$.

7.5 Tree-Cotree Algorithm