Derivation of the discrete Gaussian curvature formula

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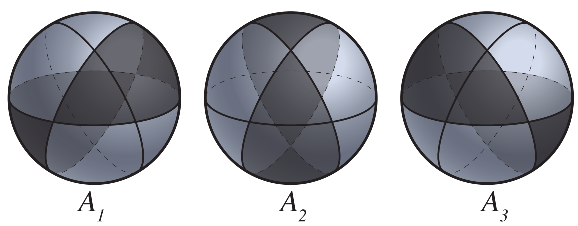
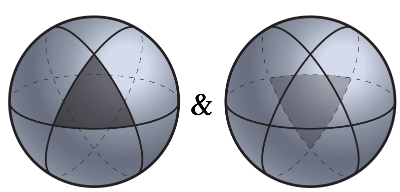
Part 1: Area of a Spherical Triangle.

Show that the area of a spherical triangle with interior angles $\alpha_1, \alpha_2, \alpha_3$ is $A = \alpha_1 + \alpha_2 + \alpha_3 - \pi$.

Consider the three diangles with angles α_i visualized below:



The important observation to make is that the intersection of *any two* diangles is simply the union of the desired triangle, Δ, and a congruent triangle. To see that the opposing triangle is congruent note that the vertices of the second triangle are simply antipodal to the vertices of the first, and thus they have the same interior angles.



We can use the inclusion-exclusion principle to express the area of triangle in terms of the areas of the diangles:

$$\text{Area}(S^2) = A_1 + A_2 + A_3 - \text{Area}(D_1 \cap D_2) - \text{Area}(D_1 \cap D_3) - \text{Area}(D_2 \cap D_3) + \text{Area}(D_1 \cap D_2 \cap D_3).$$

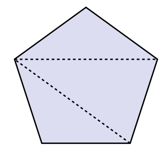
Using the observation that the area of any of the intersections is $2A$ we obtain $4(A + \pi) = A_1 + A_2 + A_3$. It remains to compute the area of a spherical diangle. A simple geometric observation is that the area of the diangle must be proportional to it's angle. Thus, since the area of the diangle with angle π is the area of the sphere 4π we conclude that the area of a diangle with angle α is simply 4α. Plugging this in yields the correct area for a spherical triangle:

$$A = \alpha_1 + \alpha_2 + \alpha_3 - \pi.$$

Part 2: Area of a Spherical Polygon.

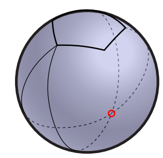
Show that the area of a spherical polygon with consecutive interior angles $\beta_1, \ldots, \beta_n$ is $A = (2-n)\pi + \sum_i \beta_i$.

The intuition is to simply triangulate the spherical polygon and then use Exercise 1.5. Assuming such a triangulation exists, it is clear that the sum of the interior angles of the triangles is nothing more than the sum of the interior angles of the original polygon. Let $T_1, \ldots, T_n$ denote the triangles in this tessellation. Using Exercise 1.5 we obtain



$$A = \sum_{i=1}^{n-2} \mathrm{Area}(T_i) = \sum_{i=1}^{n-2} \left(\sum_{j=1}^{3} \alpha_j(T_i) - \pi \right) = (2-n)\pi + \sum_{i=1}^{n} \beta_i.$$

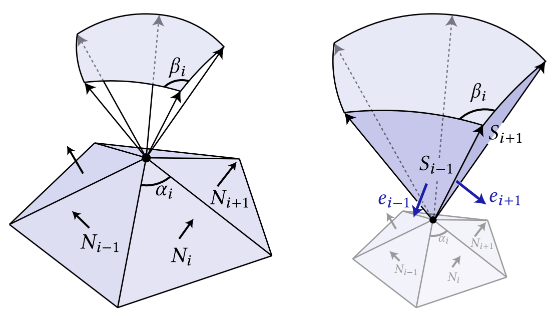
Remark 3. *Although the above answer suffices for the assignment, I do want to point out that not every spherical polygon can be triangulated! For example, consider the nonconvex spherical polygon with regular pentagonal boundary (see inset). There is only one combinatorial triangulation of the pentagon (up to symmetries), and we see that the two additional edges in the triangulations are represented by geodesics that cross! Does a similar argument still work?*



Part 3: Angle Defect.

Show that the area on the unit sphere bounded by a spherical polygon whose vertices are the unit normal around v is equal to the angle defect $d(v) = 2\pi - \sum_{f \in F_v} \angle_f(v)$, where F_v is the set of faces containing v and $\angle_f(v)$ is the interior angle of the face at vertex v.

Consider the spherical polygon whose vertices are the unit normal vectors associated to the faces adjacent to v. It will be convenient to take the center of the sphere where the spherical polygon lives to be the vertex v. The interior angles of around the vertex and the spherical polygon are denoted by α_i and β_i, respectively (see below, *left*).



Add planar faces between adjacent normals, denoted by $S_{i\pm1}$ above. The significance of these faces is that the *dihedral angle* between S_{i-1} and S_{i+1} is exactly the interior angle of the spherical polygon β_i—this is because spherical geodesics (great circles) are obtained by considering the intersection of a sphere with a plane going through the origin. Now consider the edge e_{i-1} adjacent to v which lies in the intersection of the faces f_i and f_{i-1}. This means that e_{i-1} is orthogonal to both N_i and N_{i-1}, and so e_{i-1} is orthogonal to the face S_{i-1}. Similarly, the edge e_{i+1} is orthogonal to the face S_{i+1} (see above, *right*).

Finally, notice that since e_{i-1} and e_{i+1} are the normal vectors to the planar faces S_{i-1} and S_{i+1} we conclude that the dihedral angle β_i is complementary to the angle between e_{i-1} and e_{i+1}. Since $\alpha_i = \angle(e_{i-1}, e_{i+1})$ we conclude that $\beta_i = \pi - \alpha_i$. The desired result now follows immediately:

$$d(v) = 2\pi - \sum_{f \in F_v} \angle_f(v) = 2\pi - \sum_{i=1}^{n} \alpha_i = 2\pi - \sum_{i=1}^{n} (\pi - \beta_i) = (2-n)\pi + \sum_{i=1}^{n} \beta_i = \mathrm{Area}(P).$$

The last equality follows from Part 3.

Part 4: Discrete Gauss-Bonnet Theorem.

Consider a (connected, orientable) simplicial surface K with finitely many vertices V, edges E and faces F. Show that a discrete analog of the Gauss-Bonnet theorem holds for simplicial surfaces, namely $\sum_{v \in V} d(v) = 2\pi\chi(K)$, where $\chi(K) = |V| - |E| + |F|$ is the Euler characteristic of the surface.

Since K is a simplicial surface we know that the sum of the interior angles of every face is π. Using the definition of the angle defect we see that

$$\sum_{v \in V} d(v) = \sum_{v \in V} \left(2\pi - \sum_{f \in F_v} \angle_f(v) \right) = \sum_{v \in V} 2\pi - \sum_{f \in F} \sum_{v \in V_f} \angle_f(v) = 2\pi|V| - \pi|F|.$$

Furthermore, since K is simplicial *and has no boundary* we have that $2|E| = 3|F|$. Directly substituting this in yields the discrete Gauss-Bonnet theorem

$$\sum_{v \in V} d(v) = 2\pi|V| - \pi|F| = 2\pi|V| - \pi(3-2)|F| = 2\pi\left(|V| - |E| + |F|\right) = 2\pi\chi(K).$$

Remark 4. *The discrete Gauss-Bonnet theorem can be traced back to Descartes[2]—in fact, in the polyhedral setting it should probably be referred to as the Descartes-Gauss-Bonnet theorem.*

Remark 5. *Note the relatioship between the discrete Gauss-Bonnet theorem and the interpretation of the angle defect as the area of an associated spherical polygon: assuming that the surface is convex then the Gauss map of the surface will yield a tessellation of the sphere. The area of the sphere is 4π and by the above we see that this is the same thing as the total angle defect.*

[2] *Progymnasmata de solidorum elementis* by René Descartes (1650)