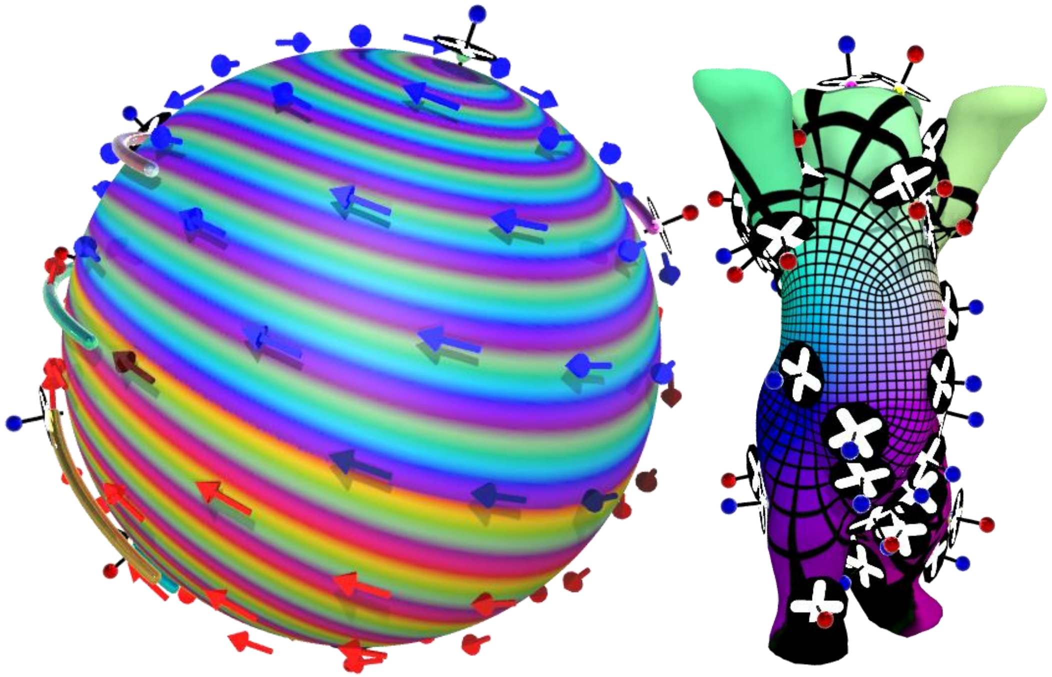
Point Vortex Dynamics on Closed Surfaces

Master thesis from Marcel Padilla
Supervisor: Prof. Dr. Ulrich Pinkall



Technische Universität Berlin

Dies Mathematicus
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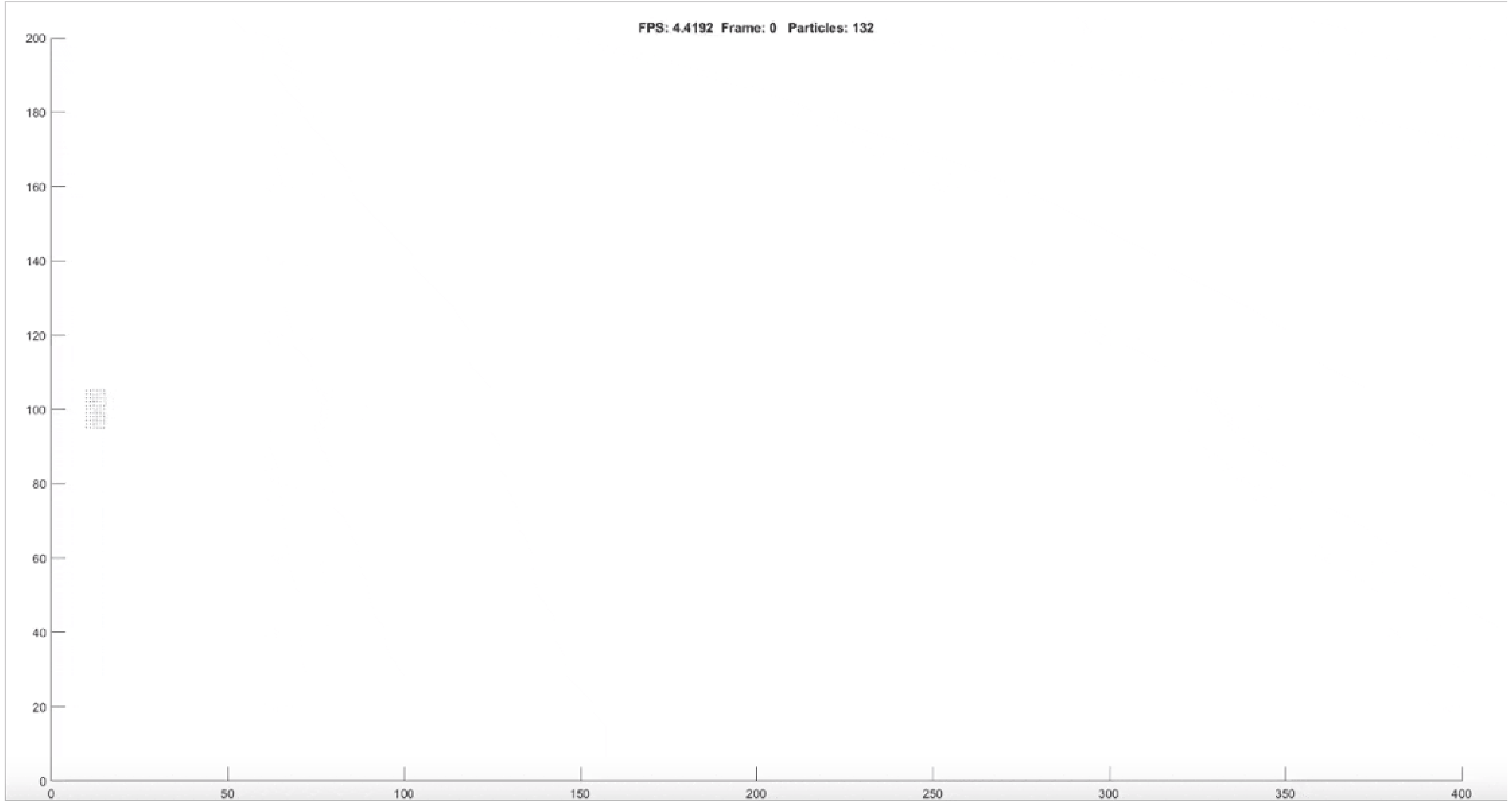
Motivation

2D fluid simulation

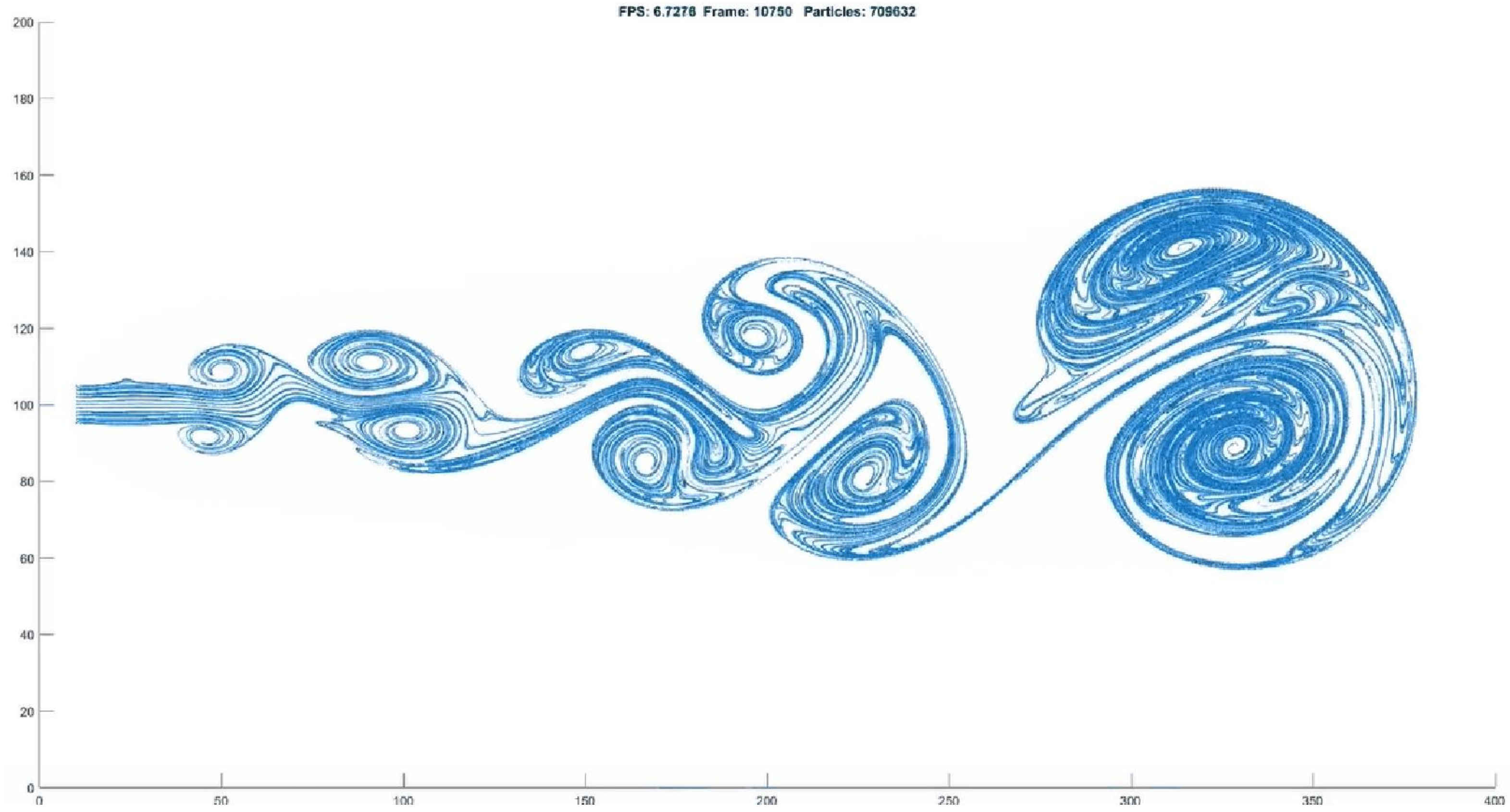


Lots of whirl!

https://www.youtube.com/watch?v=cM47L5RddsM

Motivation

2D fluid simulation



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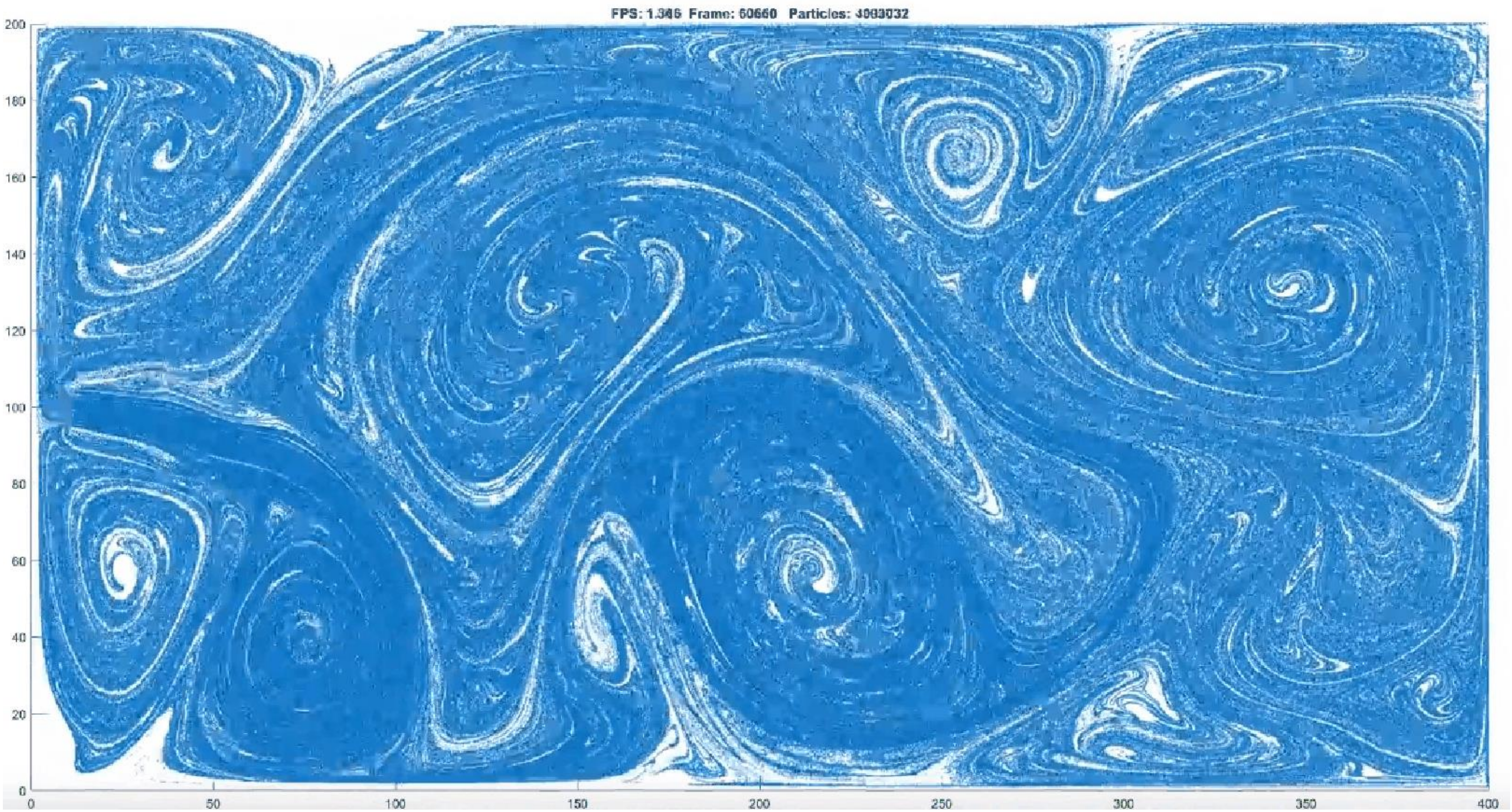

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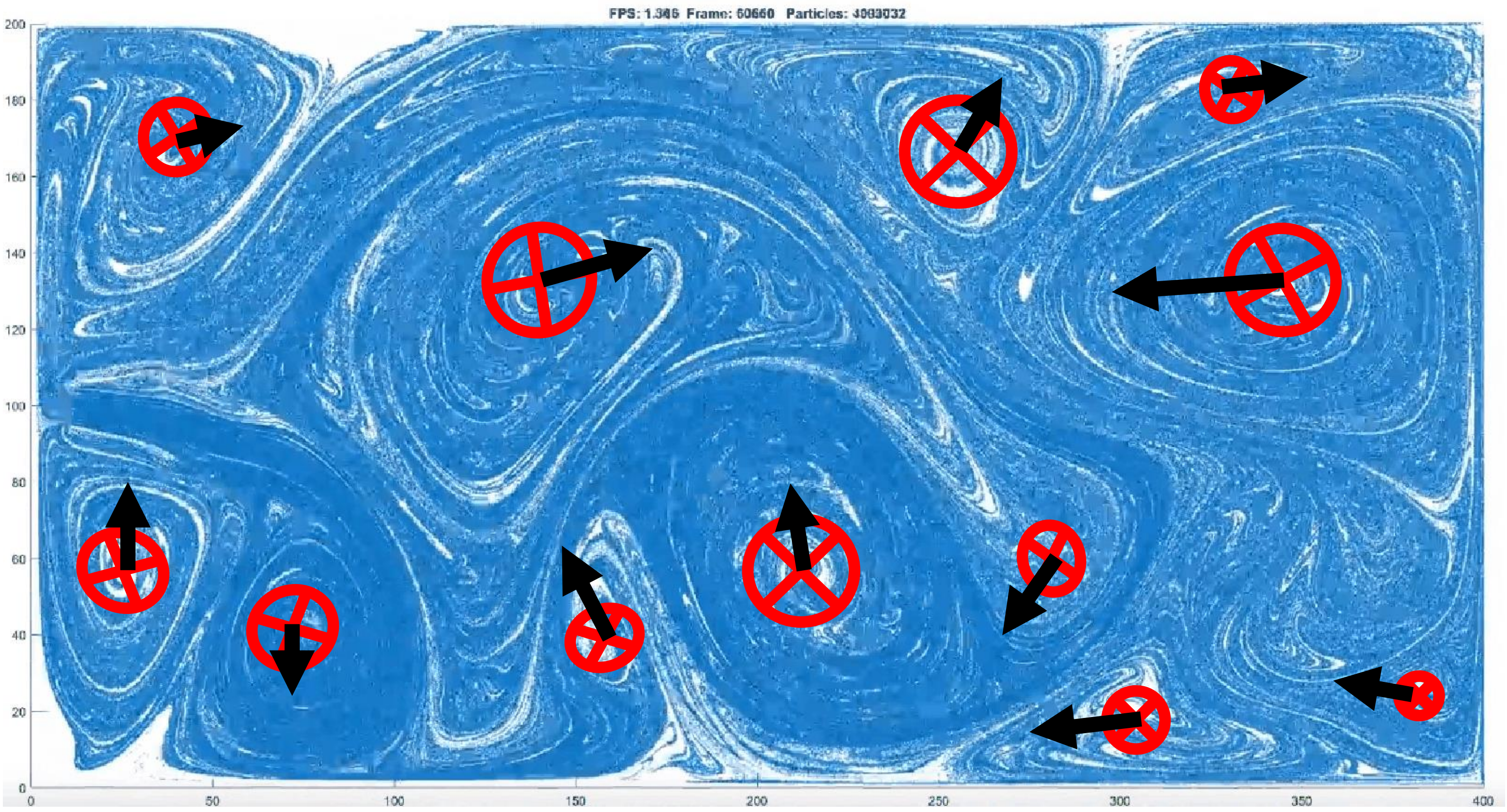
Motivation

2D fluid simulation

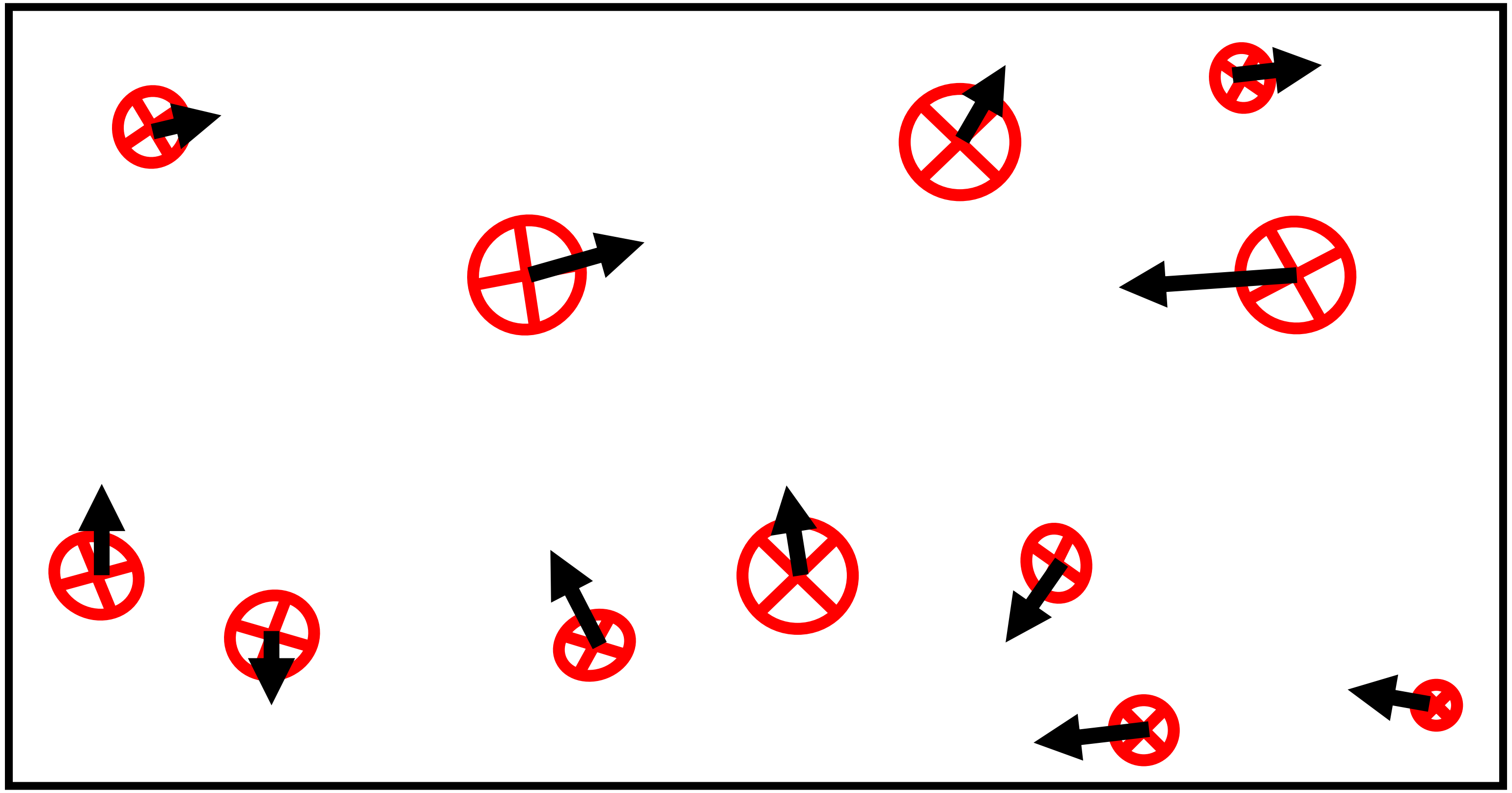


Describe the fluid by ist whirling centers

https://www.youtube.com/watch?v=cM47L5RddsM

Motivation

2D fluid simulation

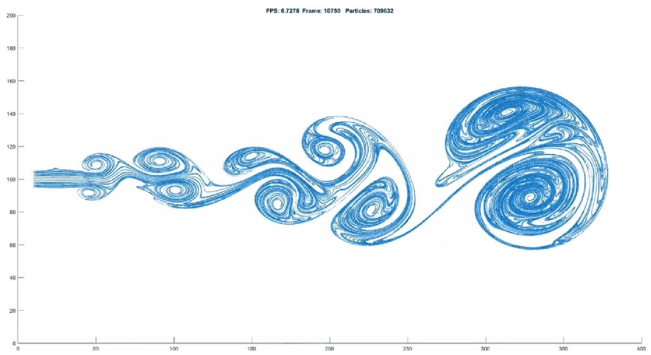


Describe the fluid by ist whirling centers

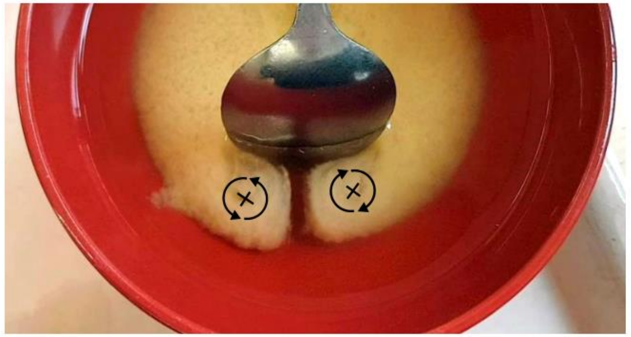
https://www.youtube.com/watch?v=cM47L5RddsM

Motivation

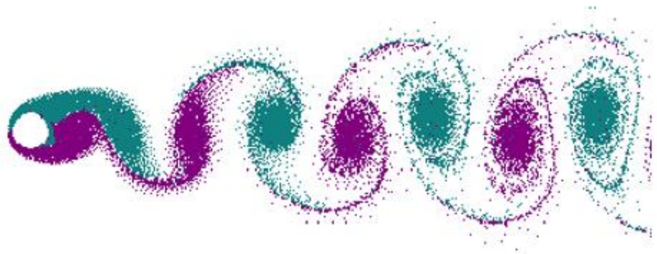
Jet propulsion



Spoon in a liquid



Kármán vortex street



Hurricanes

flat, spherical or bunny shaped earth

Jet: https://www.youtube.com/watch?v=cM47L5RddsM, Street: http://www.mcef.ep.usp.br/staff/jmeneg/cesareo/vort2.gif, Hurricanes Katia, Irma and Jose image by NASA

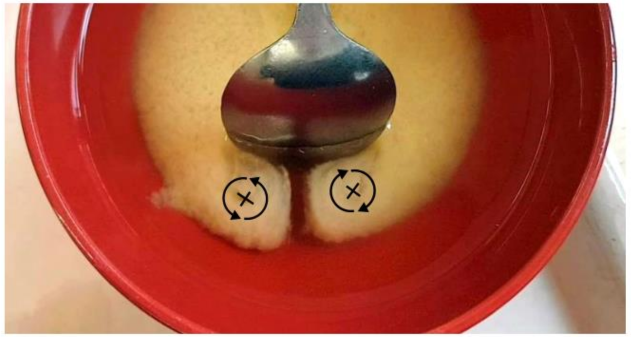
Motivation

Questions:

How can we describe 2D fluid dynamics using point vortices?

How can this theory be applied to closed surfaces of genus 0?

Spoon in a liquid



Hurricanes

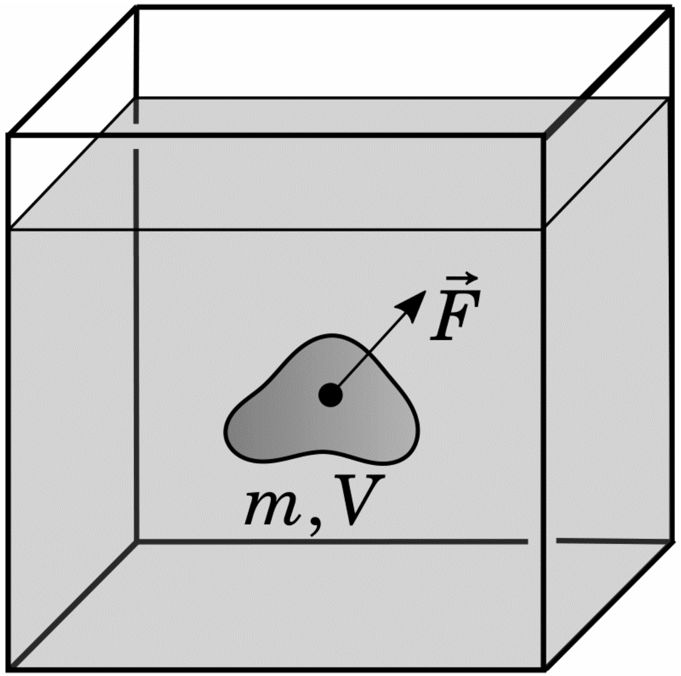
flat, spherical or bunny shaped earth

Hurricanes Katia, Irma and Jose image by NASA

Crashkurs Fluiddynamik

The dynamics of the velocity field u

$$F = ma \quad \Rightarrow \quad F = m\left(\frac{D}{Dt}u\right) \quad \Rightarrow \quad mg \; - V\nabla p \; + V\mu\Delta u = m\left(\frac{D}{Dt}u\right)$$



mg → gravity

$-V\nabla p$ → pressure

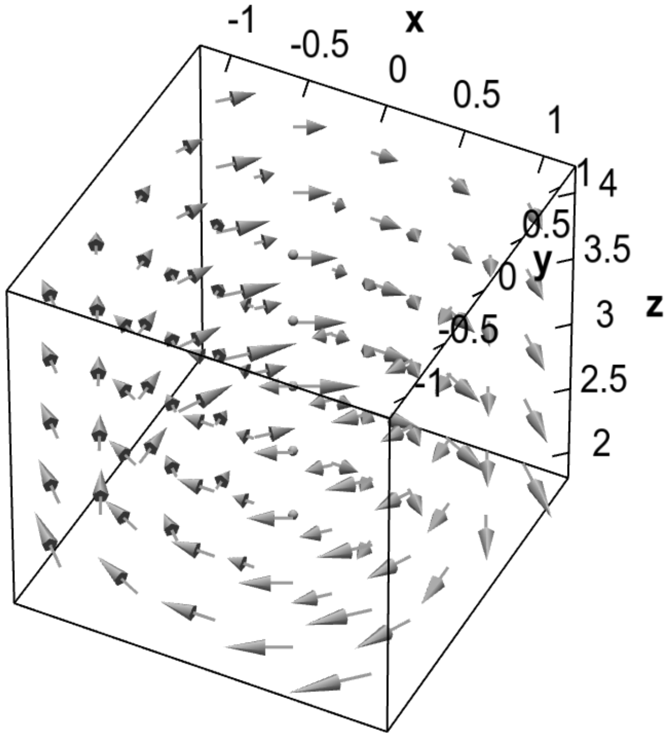
$+V\mu\Delta u$ → viscosity

Crashkurs Fluiddynamik

Navier-Stokes equations

$$\frac{\partial}{\partial t}u + (u \cdot \nabla)u + \frac{1}{\rho}\nabla p = g + \nu\Delta u$$

$$\nabla \cdot u = 0$$



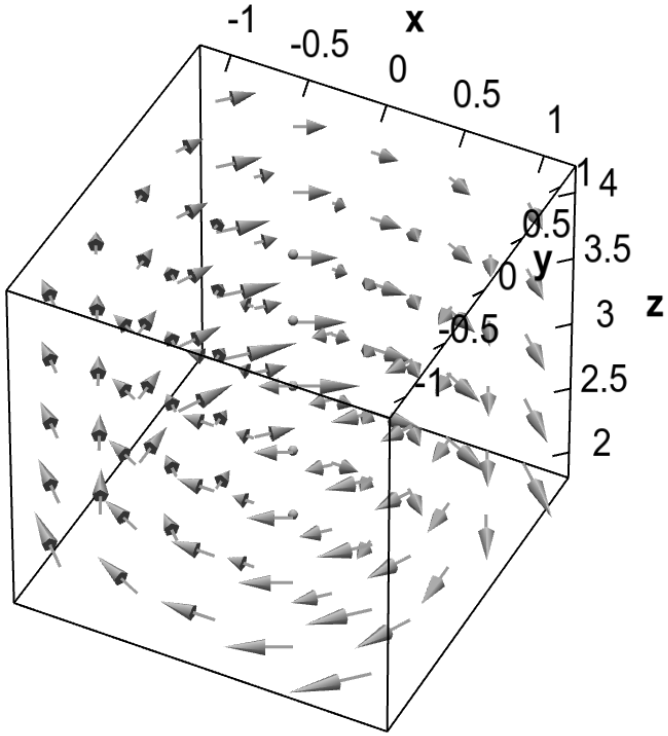
u = velocity
ρ = desnity
g = gravity
ν = dynamic viscosity

Crashkurs Fluiddynamik

Euler equations

$$\frac{\partial}{\partial t}u + (u \cdot \nabla)u + \frac{1}{\cancel{\rho}}\nabla p = \cancel{g} + \cancel{\nu\Delta u}$$

$$\nabla \cdot u = 0$$



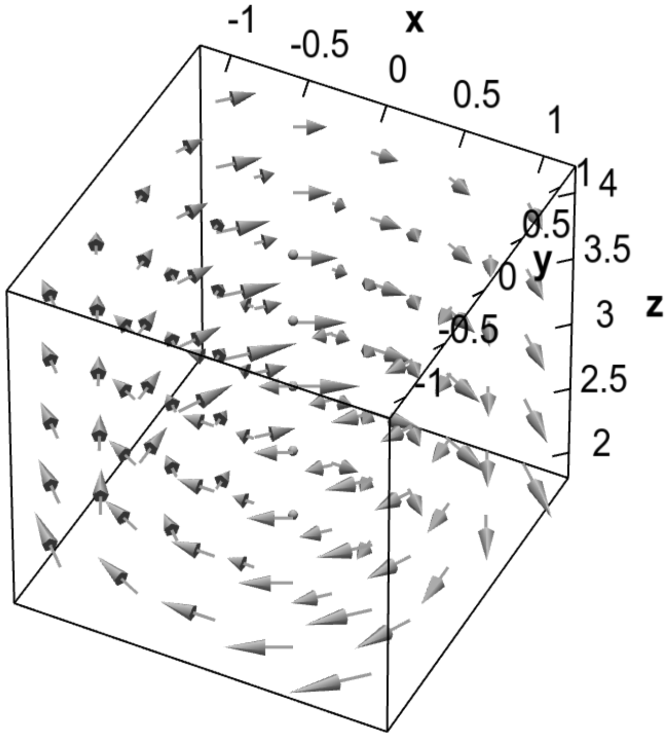
u = velocity
ρ = 1 constant density
g = 0 no gravity
ν = 0 no viscosity

Crashkurs Fluiddynamik

Euler equations

$$\frac{\partial}{\partial t}u + (u \cdot \nabla)u = -\nabla p$$

$$\nabla \cdot u = 0$$



u = velocity

ρ = 1 constant density

g = 0 no gravity

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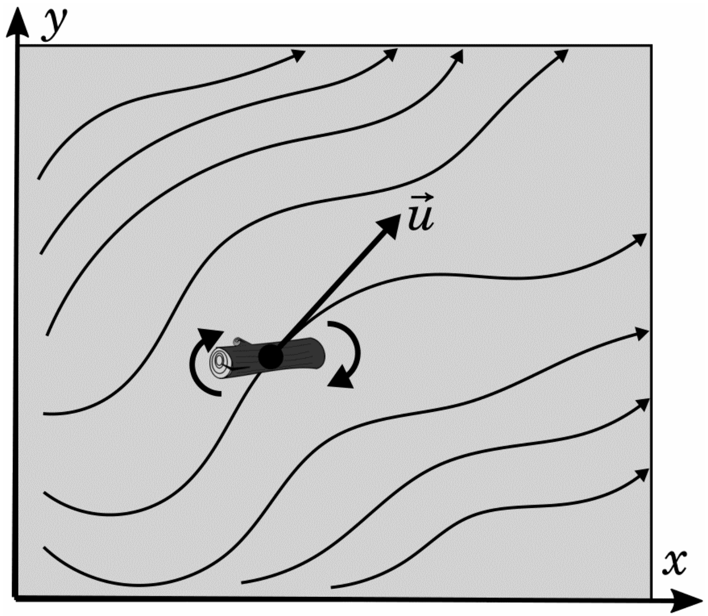
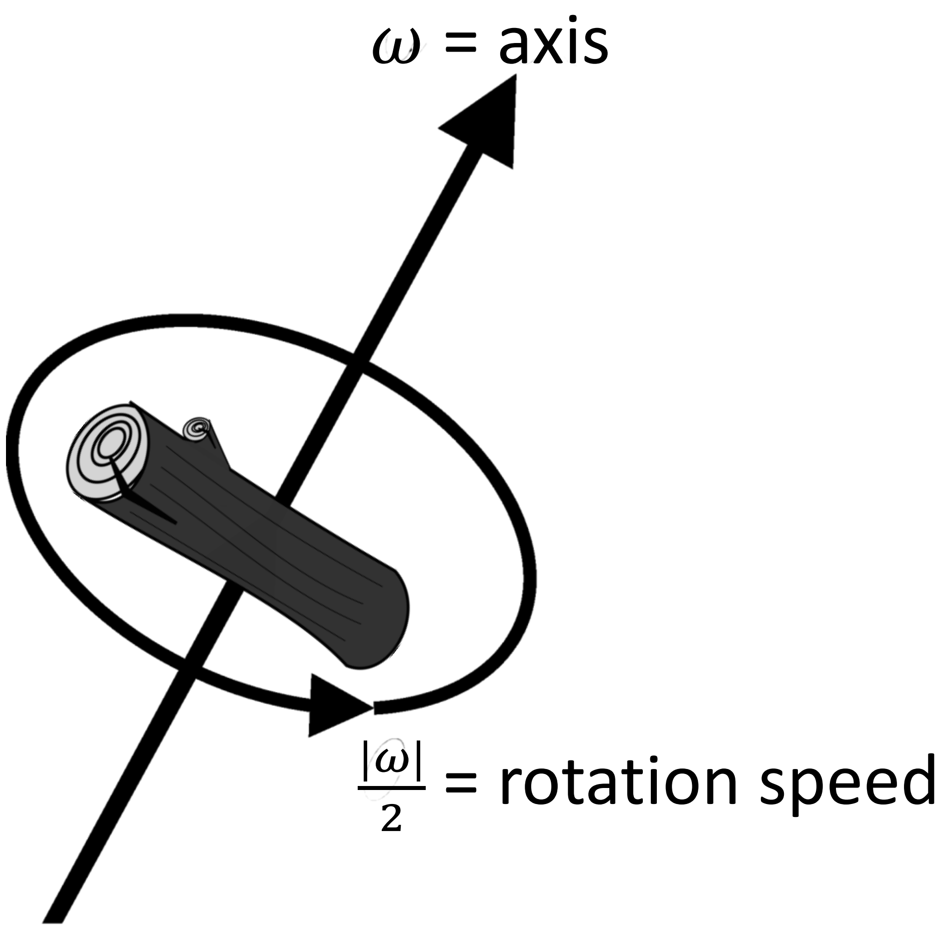
https://mathinsight.org/vector_field_overview

Fluid Vorticity

Definition: Vorticity

$$\omega := \nabla \times u = \begin{pmatrix} \frac{\partial}{\partial z}u_y - \frac{\partial}{\partial y}u_z \\ \frac{\partial}{\partial z}u_x - \frac{\partial}{\partial x}u_z \\ \frac{\partial}{\partial x}u_y - \frac{\partial}{\partial y}u_x \end{pmatrix}$$

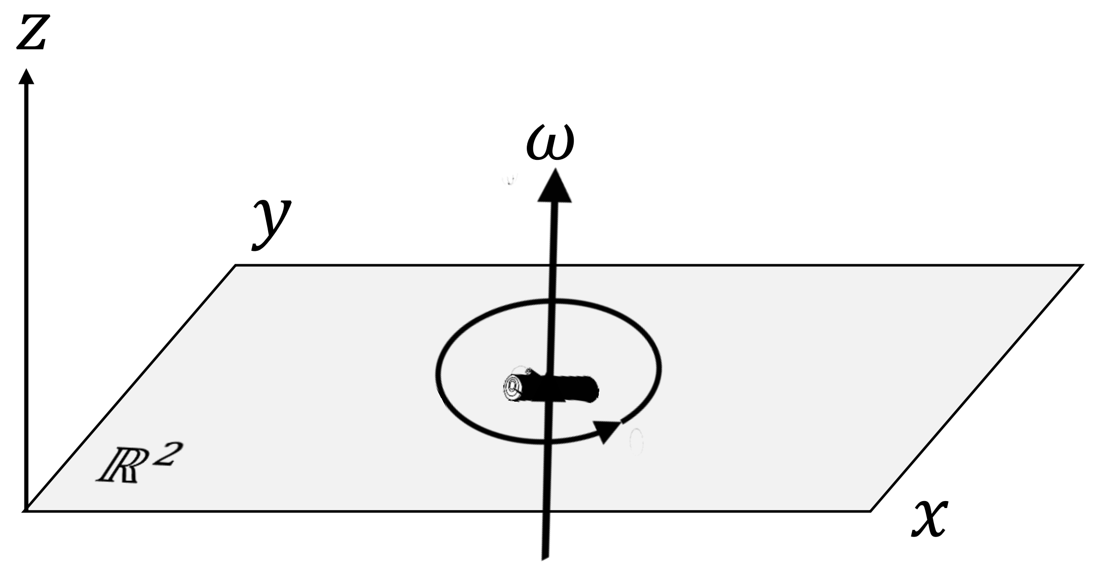




Fluid Vorticity

In 2D:

Rotation axis is normal to the surface



Definition: scalar vorticity

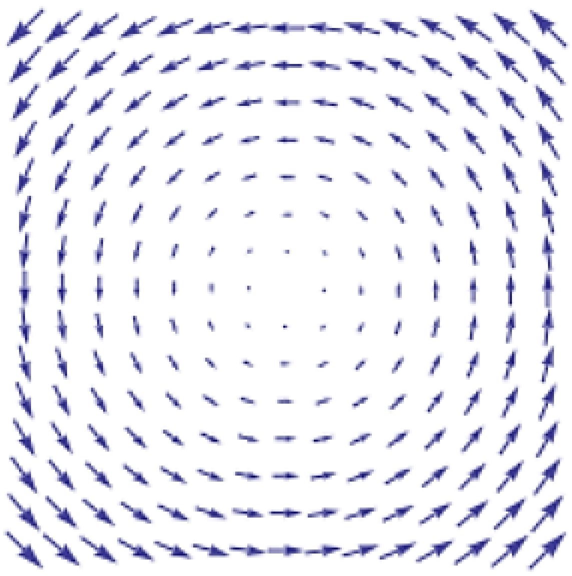
$$\omega_{skalar} := \omega \cdot n$$

Example in $\mathbb{R}^2$:

$$\nabla \times u = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial}{\partial x}u_y - \frac{\partial}{\partial y}u_x \end{pmatrix} = \omega_{skalar} n$$

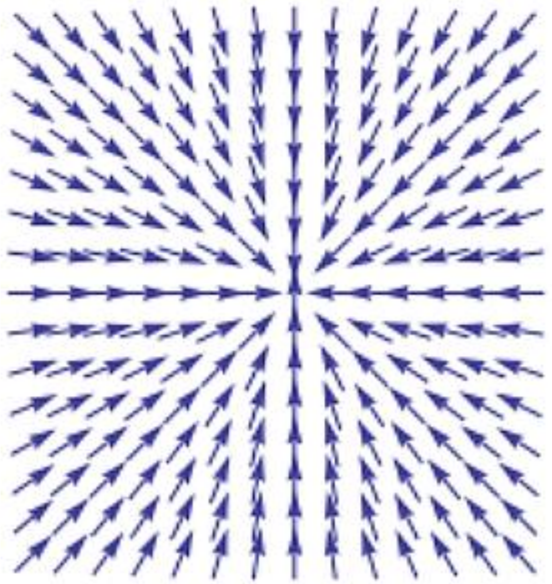
Fluid Vorticity

Only whirl



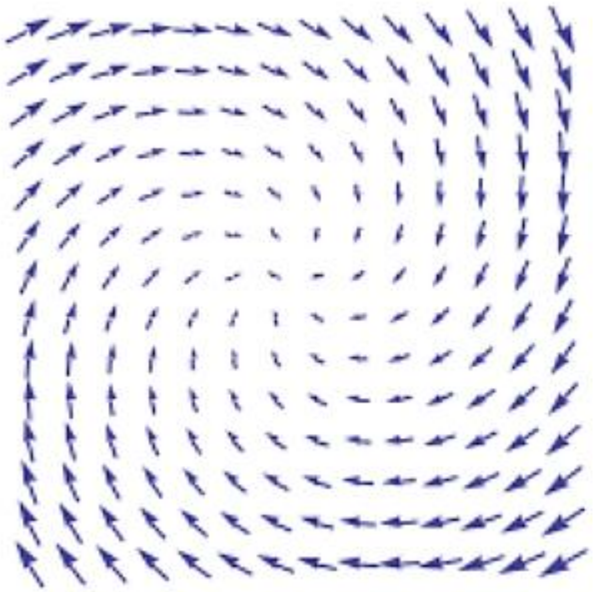
$$u = \begin{pmatrix} -y \\ x \end{pmatrix}, \omega \equiv 2$$

Potential field



$$u = \nabla f, \omega \equiv 0$$

Whirl + potential field



$$u = \begin{pmatrix} y \\ -x \end{pmatrix} + \nabla h, \omega \equiv -2$$

Fluid Vorticity

Euler equations

$$\frac{\partial}{\partial t}u + (u \cdot \nabla)u = -\nabla p$$

+

Curl

$$\nabla \times (\cdot)$$

=

Vorticity equation

$$\frac{\partial}{\partial t}\omega + u \cdot \nabla\omega = \omega \cdot \nabla u$$

$= 0$ in 2D

[ETK07]

Fluid Vorticity

Euler equations

$$\frac{\partial}{\partial t}u + (u \cdot \nabla)u = -\nabla p$$

$+$

Curl

$$\nabla \times (\cdot)$$

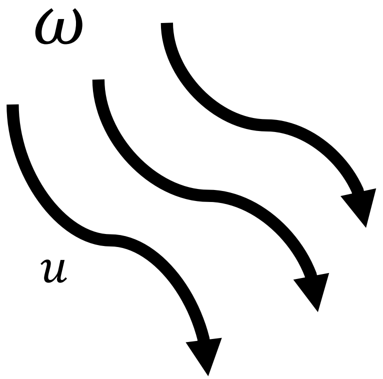
$=$

Vorticity equation

$$\frac{\partial}{\partial t}\omega + u \cdot \nabla\omega = 0$$

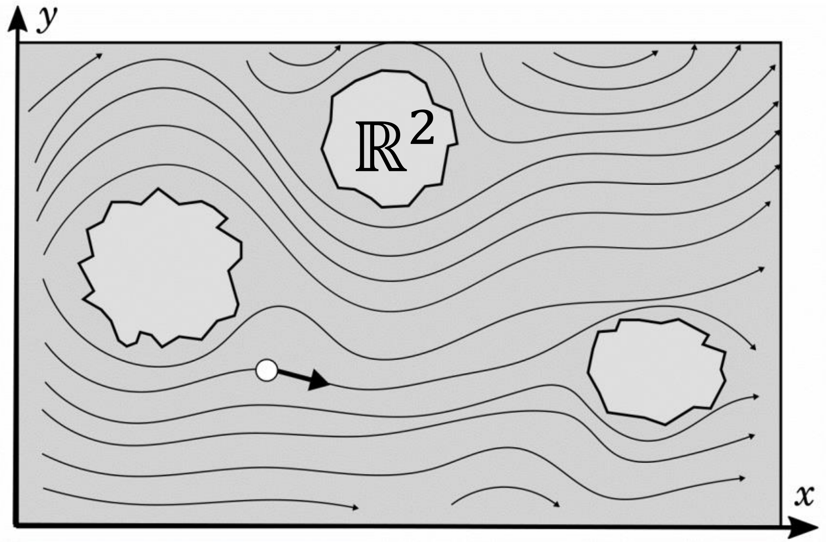
$\Rightarrow$

Transport by u



[ETK07]

Stream Function



Definition: ψ is 2D stream function of u if

$$\psi: M \mapsto \mathbb{R}$$

$$\nabla^S \psi := n \times \nabla\psi = u$$

Theorem: $\psi(x) = \int_M G(x,p)\omega(p)dp$, $G(x,y)$ = Green‘s function on M

Stream Function

Theorem: $\psi(x) = \int_M G(x,p)\omega(p)dp,\quad G(x,y) =$ Green's function on M

$$\nabla^s\psi = \mathrm{n}\times\nabla\psi = u \;\longrightarrow\; \omega = \nabla\times u = \nabla\times(\mathrm{n}\times\nabla\psi) = \Delta\psi$$

→ Poisson problem

$$\omega = \Delta\psi$$

Trick: Green's function

$$G: M^2\backslash\{(x,x)\mid x\in M\} \mapsto \mathbb{R}$$

$$\Delta_x G(x,y) = \delta(x-y)$$

Stream Function

Theorem: $\psi(x) = \int_M G(x,p)\omega(p)dp, \quad G(x,y) =$ Green's function on M

$$\Delta_x\psi(x) = \Delta_x \int_M G(x,p)\omega(p)dp$$

$$= \int_M \Delta_x G(x,p)\omega(p)dp$$

$$= \int_M \delta(x-p)\omega(p)dp$$

$$= \omega(x)$$

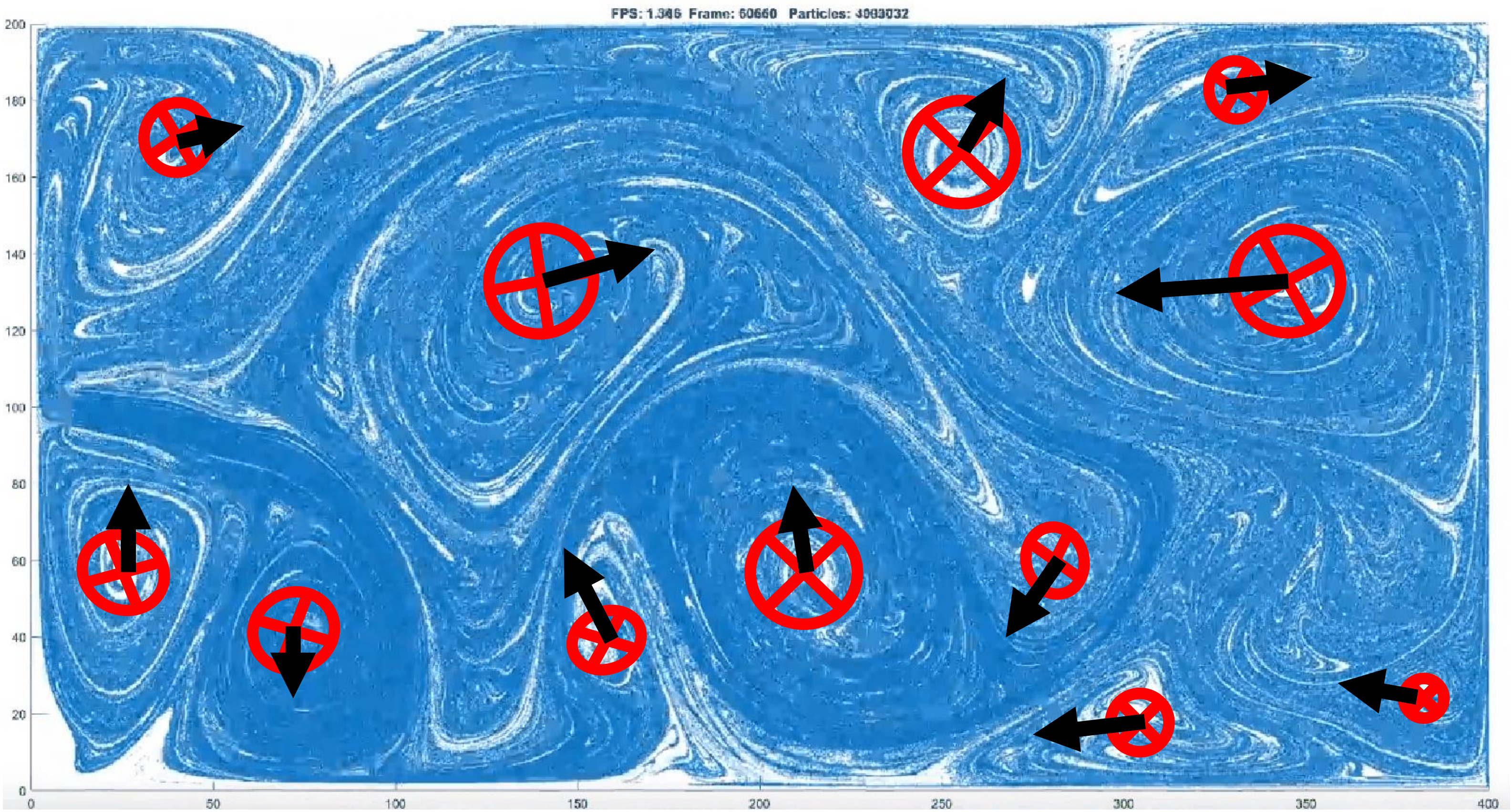
$$\Delta_x G(x,y) = \delta(x-y)$$

→ Poisson problem

$$\omega = \Delta\psi$$

Point Vortices Model

2D fluid simulation

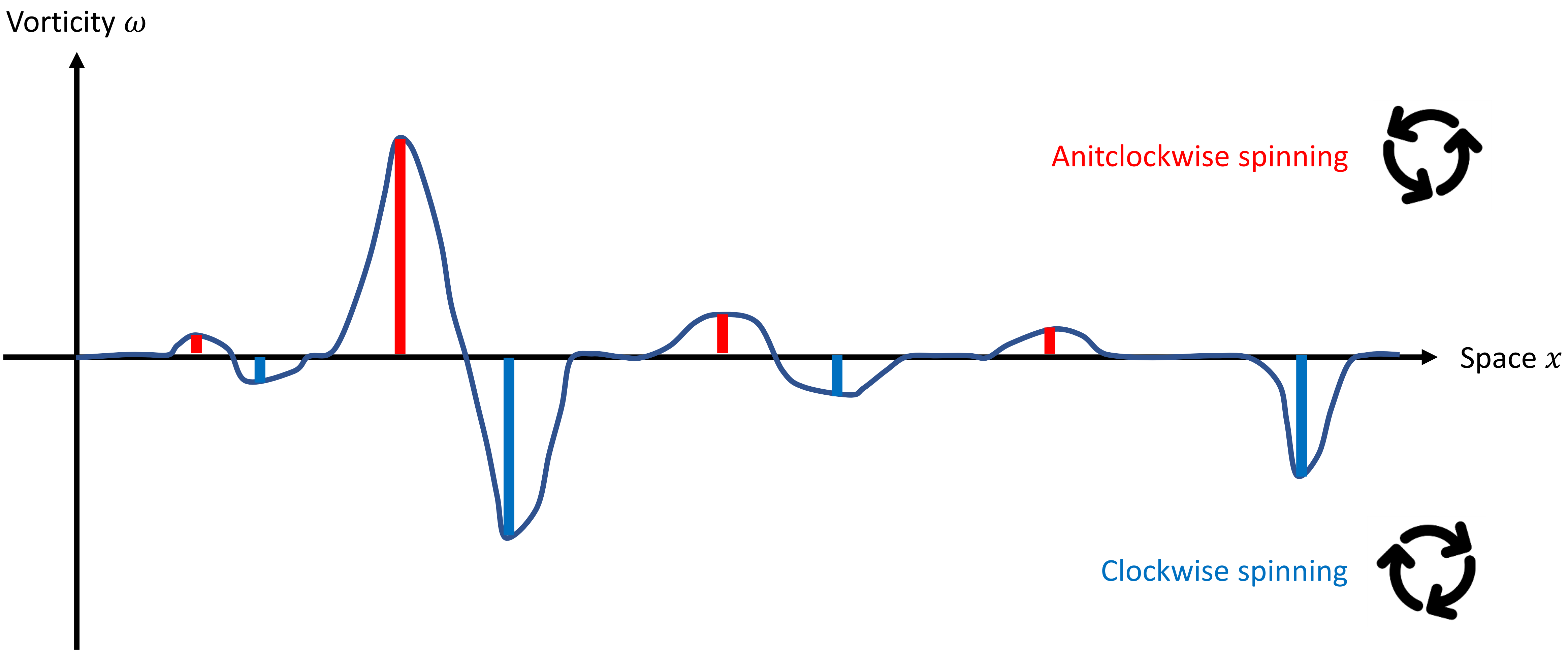


Punktwirbel
$p_1, \ldots, p_n \in M$
mit Wirbelstärken
$\omega_1, \ldots, \omega_n \in \mathbb{R}$

So… what is ω like?

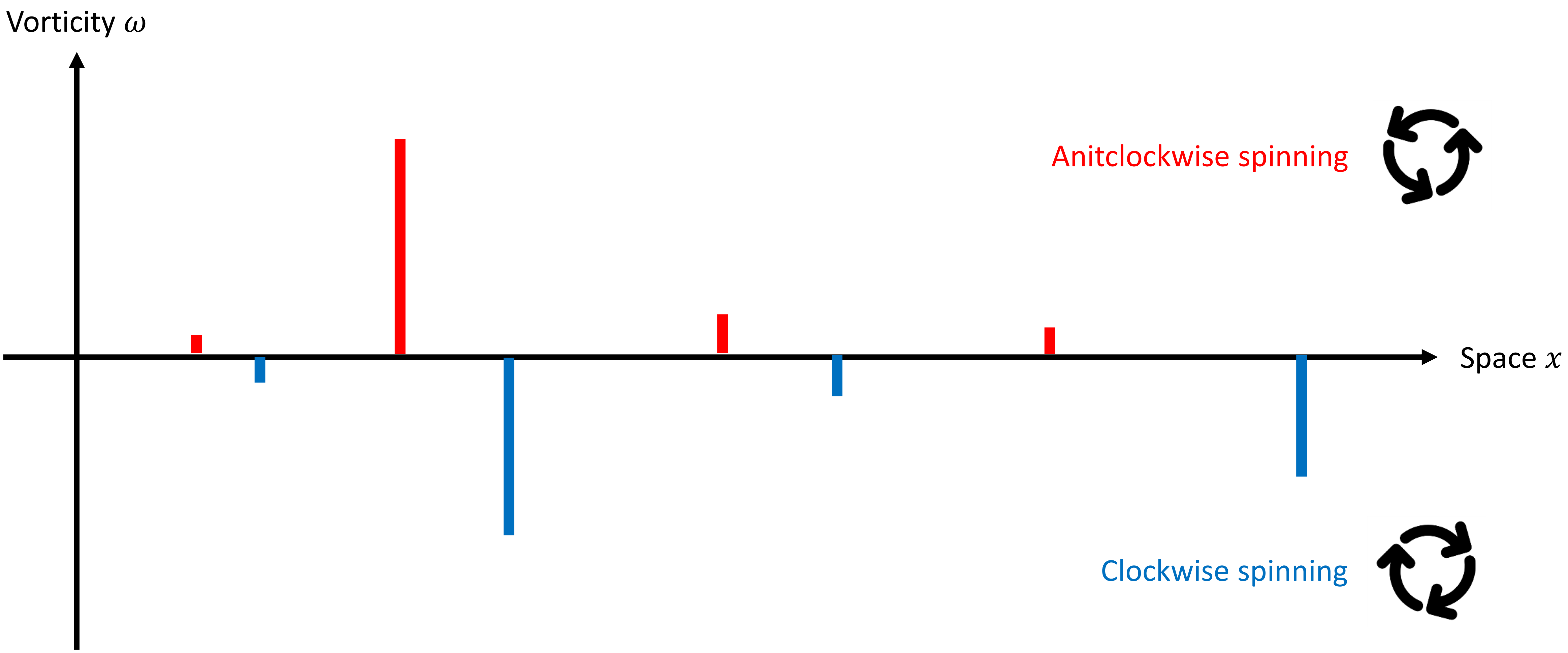
Point Vortices Model

Primitive approximation of vorticity



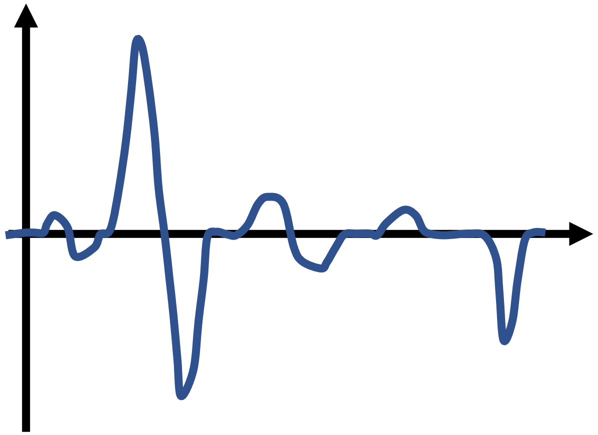
Point Vortices Model

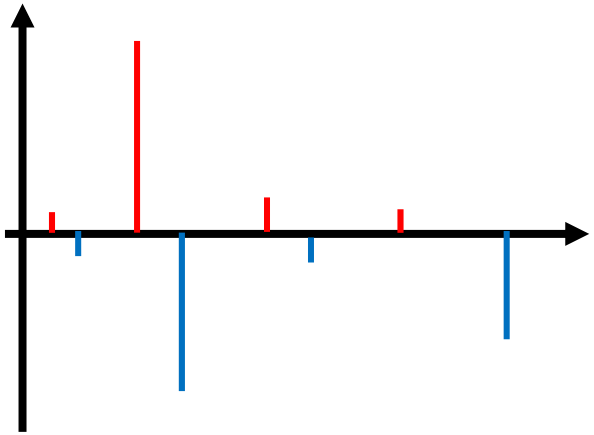
Primitive approximation of vorticity



Point Vortices Model

Primitive approximation of vorticity





Smooth

$$\omega(x) \in C^\infty(\mathbb{R}^2)$$

Discrete concentrated

$$\omega(x) = \sum_{i=1}^{n} \omega_i \delta(x - p_i)$$

Point Vortices Model

Unite!

$$\nabla^S_x \psi(x) = u(x) \quad + \quad \psi(x) = \int_M G(x,p)\omega(p)dp \quad + \quad \omega(x) = \sum_{i=1}^{n} \omega_i \delta(x - p_i)$$

To get this:

$$u(x) = \nabla^S_x \psi(x)$$

Point Vortices Model

Unite!

$$\nabla_x^S \psi(x) = u(x) \quad + \quad \psi(x) = \int_M G(x,p)\omega(p)dp \quad + \quad \omega(x) = \sum_{i=1}^{n} \omega_i \delta(x - p_i)$$

To get this:

$$u(x) = \nabla_x^S \int_M G(x,p)\omega(p)dp$$

Point Vortices Model

Unite!

$$\nabla_x^S \psi(x) = u(x) \quad + \quad \psi(x) = \int_M G(x,p)\omega(p)dp \quad + \quad \omega(x) = \sum_{i=1}^{n} \omega_i \delta(x - p_i)$$

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Point Vortices Model

Unite!

$$\nabla^S_x \psi(x) = u(x) \quad + \quad \psi(x) = \int_M G(x,p)\omega(p)dp \quad + \quad \omega(x) = \sum_{i=1}^{n} \omega_i \delta(x - p_i)$$

To get this:

$$u(x) = \int_M \nabla^S_x G(x,p) \sum_{i=1}^{n} \omega_i \delta(x - p_i) \; dp$$

Point Vortices Model

Unite!

$$\nabla^S_x \psi(x) = u(x) \quad + \quad \psi(x) = \int_M G(x,p)\omega(p)dp \quad + \quad \omega(x) = \sum_{i=1}^{n} \omega_i \delta(x - p_i)$$

To get this:

$$u(x) = \sum_{i=1}^{n} \omega_i \int_M \nabla^S_x G(x,p)\delta(x - p_i)dp$$

Point Vortices Model

Unite!

$$\nabla_x^S \psi(x) = u(x) \quad + \quad \psi(x) = \int_M G(x,p)\omega(p)dp \quad + \quad \omega(x) = \sum_{i=1}^{n} \omega_i \delta(x - p_i)$$

To get this:

$$u(x) = \sum_{i=1}^{n} \omega_i \nabla_x^S G(p_i, p)$$

Point Vortices Model

Unite!

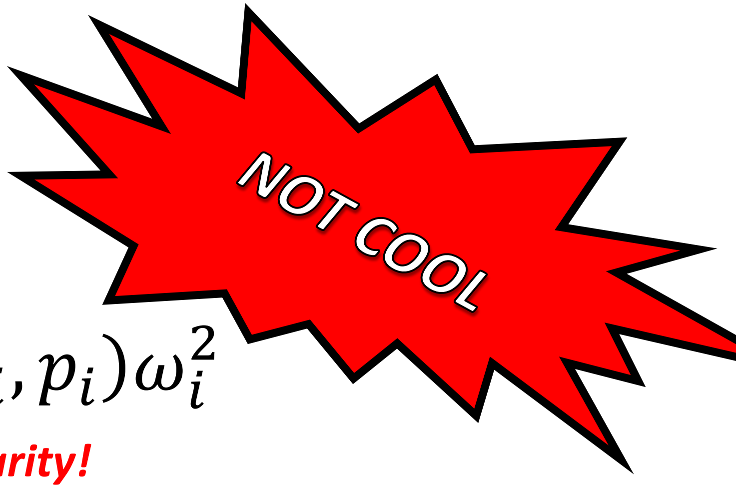
$$\nabla_x^S \psi(x) = u(x) \quad + \quad \psi(x) = \int_M G(x,p)\omega(p)dp \quad + \quad \omega(x) = \sum_{i=1}^{n} \omega_i \delta(x - p_i)$$

To get this:

$$u(x) = \sum_{i=1}^{n} \nabla_x^S G(x, p_i)\omega_i$$

Kirchhoff‘s Assumption

$$H = E_{kin} = \frac{1}{2}\int_M m|u(x)|^2 dx = -\frac{1}{2}\int_M \psi(x)\omega(x)dx$$



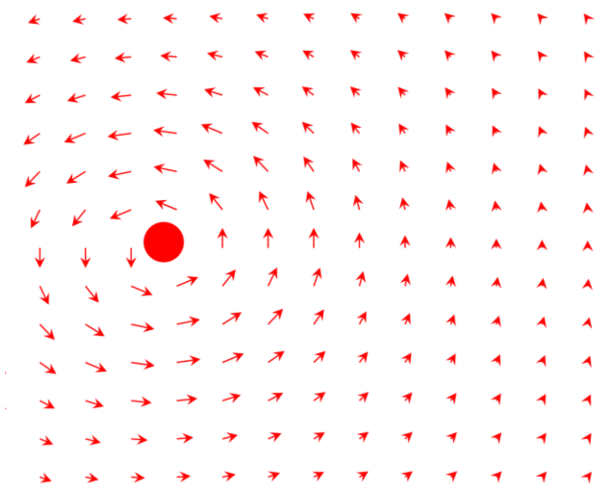
$$\Rightarrow \quad H = -\frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{n} G(p_j,p_i)\omega_i\omega_j + \sum_{i=1}^{n} G(p_i,p_i)\omega_i^2$$

Singularity!

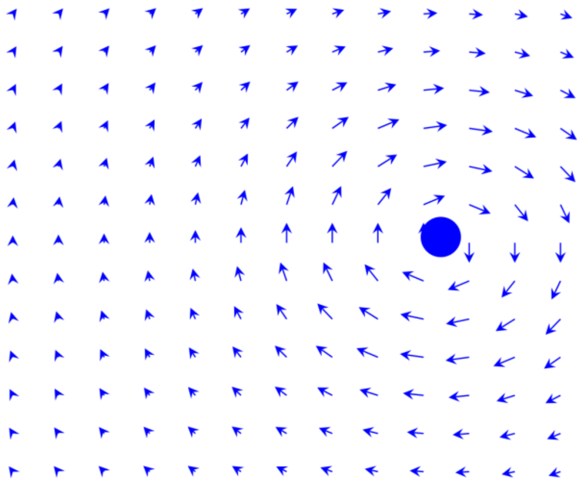
Kirchhoff‘s assumption: No selfinduction!

Kirchhoff's Assumption

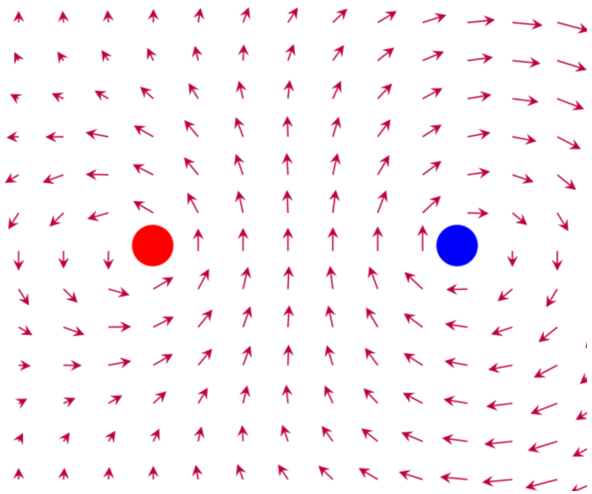
$$\forall x \notin \{p_1, \dots, p_n\}: \; u(x) = \sum_{i=1}^{n} \nabla_x^S G(x, p_i)\omega_i$$



Vortex 1



Vortex 2



Vortex 1 + Vortex 2

Vorticity equation

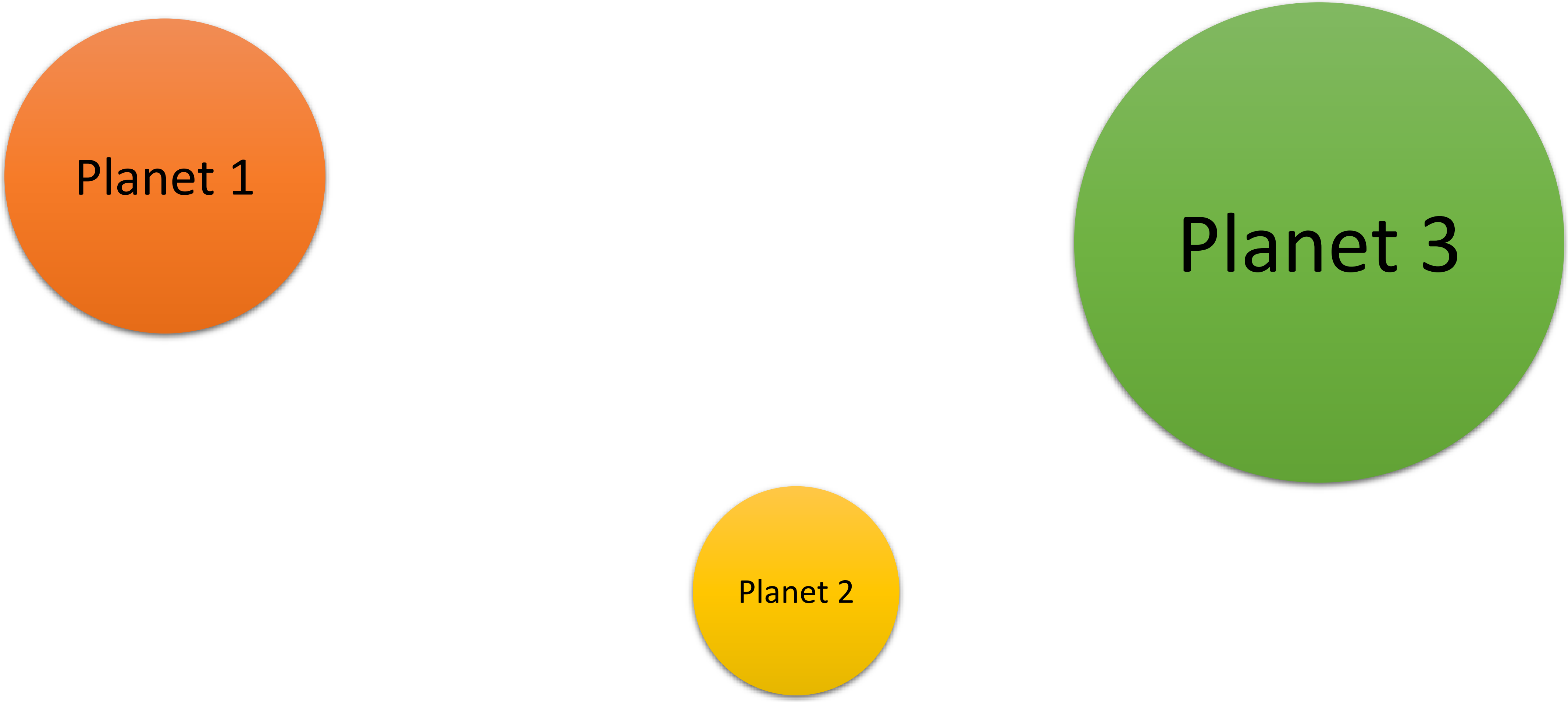
$$\frac{\partial}{\partial t}\omega + u \cdot \nabla\omega = 0$$

Transport ω by u

$$\rightarrow \quad u(p_j) = \sum_{\substack{i=1 \\ i \neq j}}^{n} \nabla_x^S G(p_j, p_i)\omega_i$$

Kirchhoff‘s Assumption

Gravitational analogy of Kichhoff‘s assumption



Kirchhoff‘s Assumption

Gravitational analogy of Kichhoff‘s assumption

Point mass!

Planet 1

Planet 3

$$F(p_1) = \sum_{i=1}^{3} G \frac{m_1 m_i}{d(p_1, p_i)^2}$$

Planet 2

Kirchhoff‘s Assumption

Gravitational analogy of Kichhoff‘s assumption

Point mass!

Planet 1

Planet 3

$$F(p_1) = \cancel{G\frac{m_1 m_1}{d(p_1,p_1)^2}} + G\frac{m_1 m_2}{d(p_1,p_2)^2} + G\frac{m_1 m_3}{d(p_1,p_3)^2}$$

Singularity!
No self induction!

Planet 2

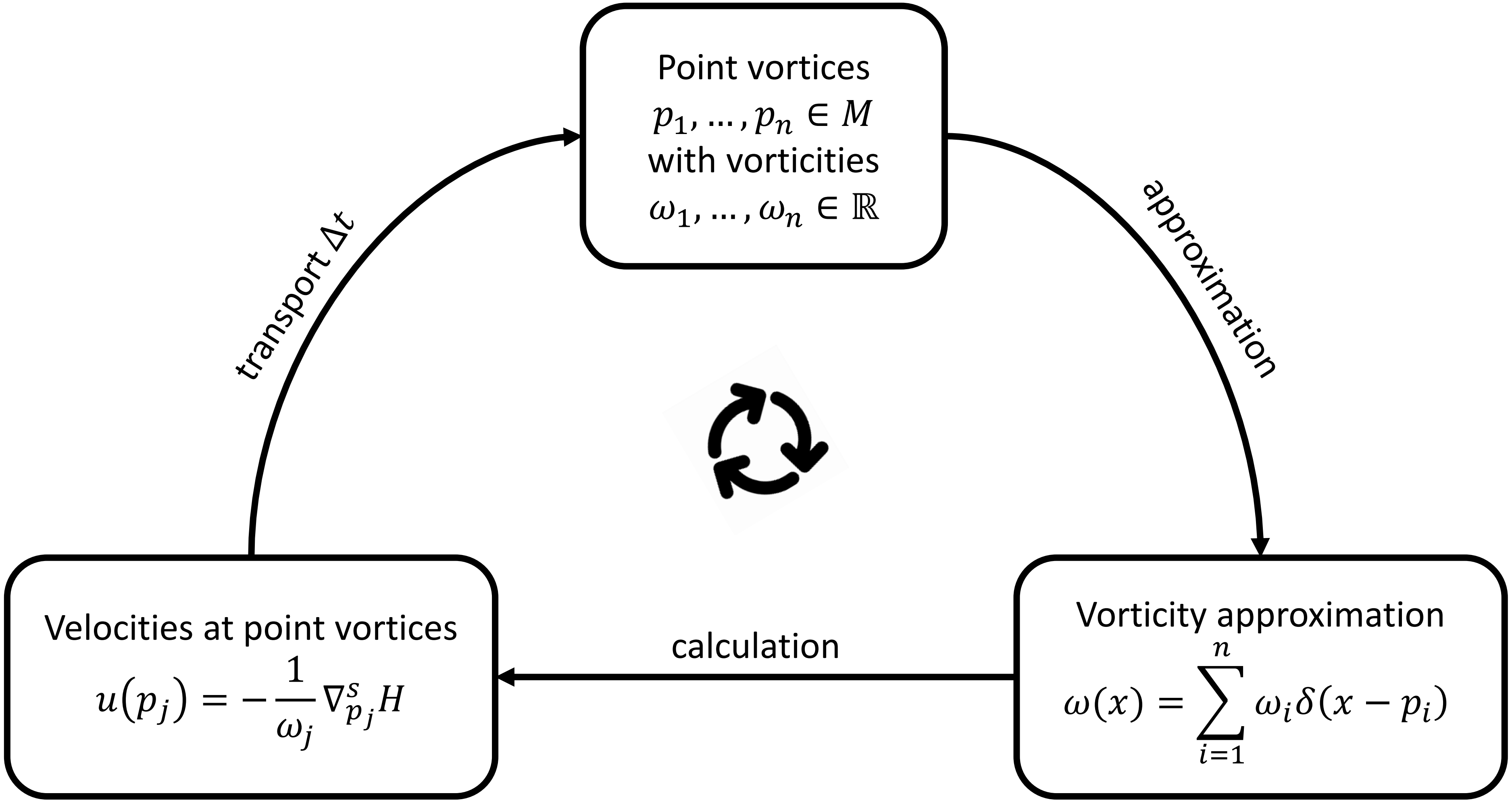
Kirchhoffs Assumption

$$H = -\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} G(x,p_i)\omega_i\omega_j + \sum_{i=1}^{n} G(p_i,p_i)\omega_i^2$$

Now just plug in G and calculate!

$$u(p_j) = -\frac{1}{\omega_j}\nabla^S_{p_j} H = \sum_{\substack{i=1\\ i\neq j}}^{n} \nabla^S_{p_j} G(p_j,p_i)\omega_i$$

Point Vortices Model



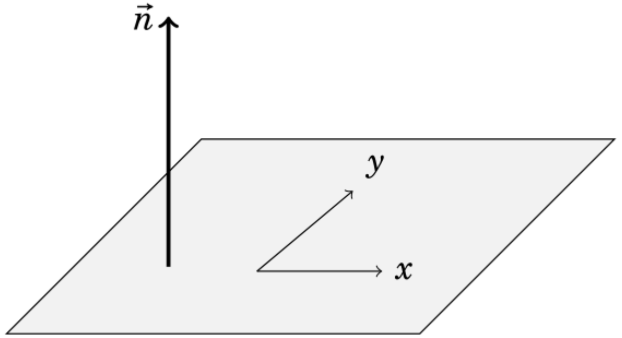
Green‘s Functions

Formula:

$$u(p_j) = -\frac{1}{\omega_j}\nabla^S_{p_j} H$$

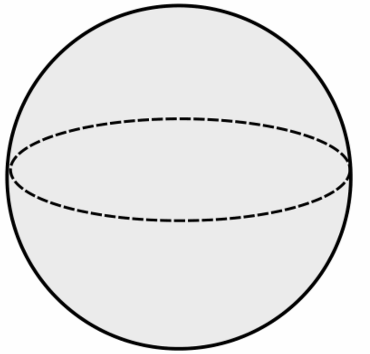
Green‘s functions examples:

Plane $\mathbb{R}^2$



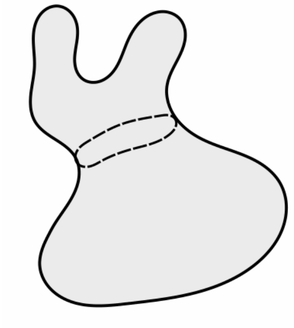
$$G(x,y) = -\frac{1}{2\pi}\ln(|x-y|)$$

Sphere S^2



$$G(x,y) = -\frac{1}{2\pi}\ln\left(\sin\left(\frac{1}{2}d_{S^2}(x,y)\right)\right)$$

Closed Surface M genus 0

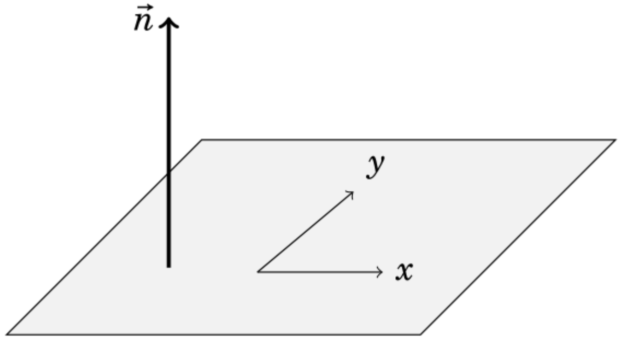


$$G(x,y) = ???$$

Not clear!

The Plane $\mathbb{R}^2$

Plane $\mathbb{R}^2$



$$G(x,y) = -\frac{1}{2\pi}\ln(|x-y|)$$

$$\nabla^s_x G(x,y) = \frac{1}{2\pi}\frac{n\times(x-y)}{|\mathrm{x}-\mathrm{y}|^2}$$

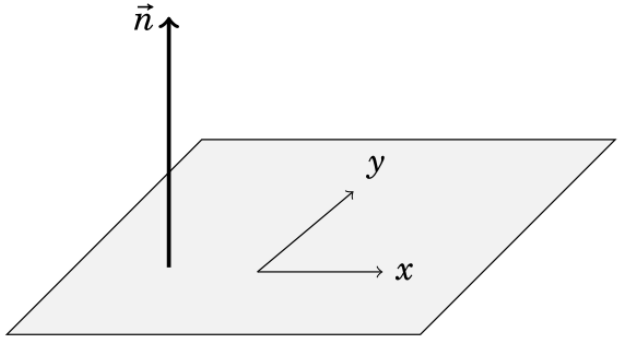
Point vortex dynamics on $\mathbb{R}^2$:

$$u(p_j) = \frac{1}{2\pi}\sum_{\substack{i=1\\ i\neq j}}^{n} \omega_i \frac{n\times(x-y)}{|\mathrm{x}-\mathrm{y}|^2}$$

[KHP91] G. Kirchhoff, K. Hensel, and M. Planck. *Vorlesungen über mathematische Physik*

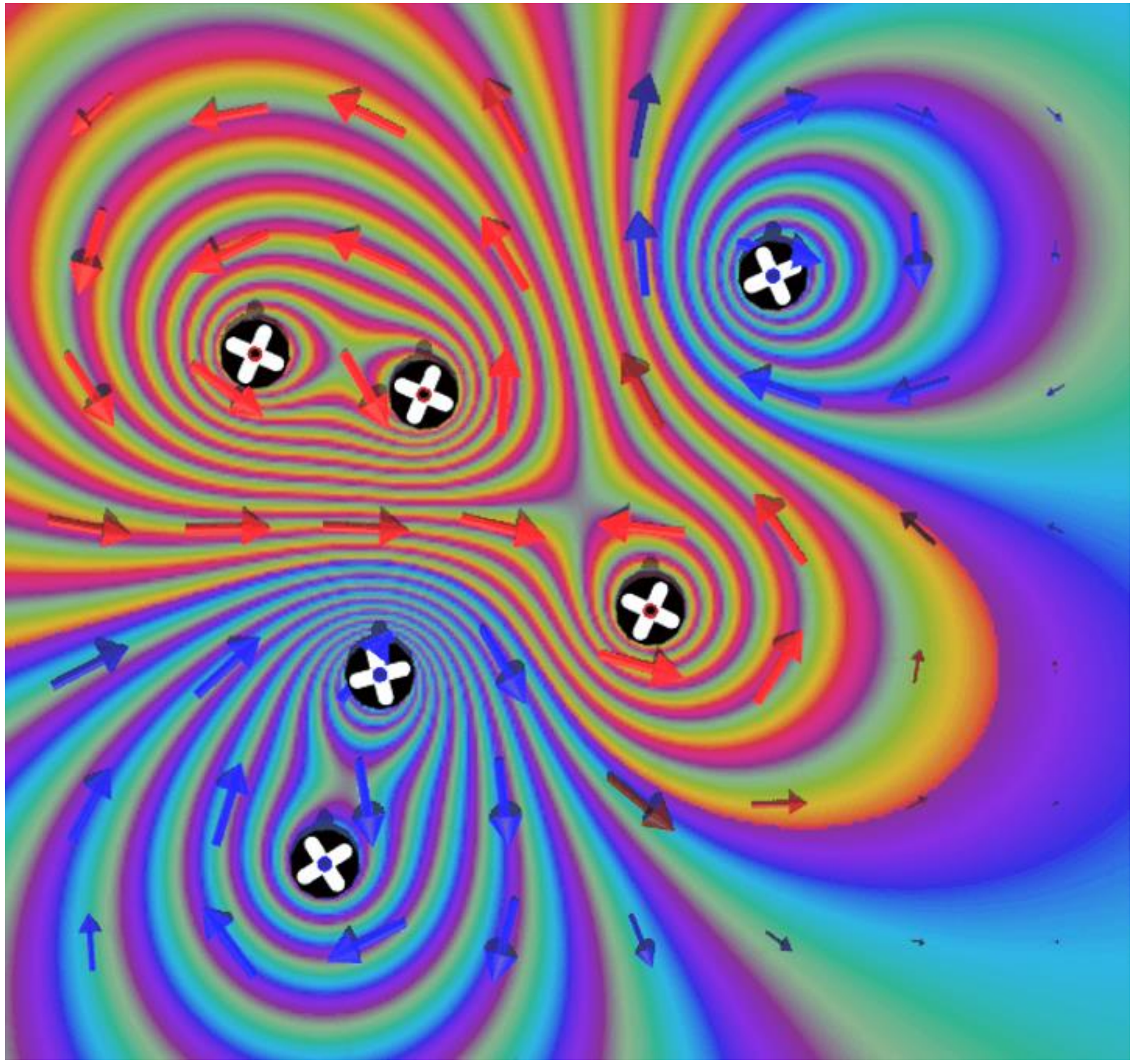
The Plane $\mathbb{R}^2$

Plane $\mathbb{R}^2$



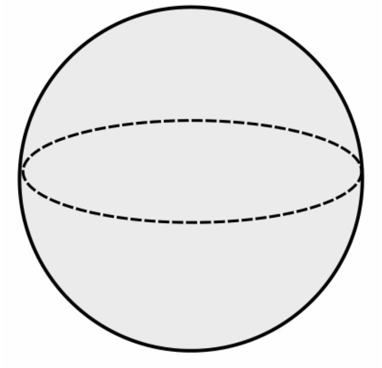
$$G(x,y) = -\frac{1}{2\pi}\ln(|x-y|)$$

$$\nabla_x^s G(x,y) = \frac{1}{2\pi}\frac{n \times (x-y)}{|x-y|^2}$$



The Sphere S^2

Sphere S^2



$$G(x,y) = -\frac{1}{2\pi}\ln\left(\sin\left(\frac{1}{2}d_{S^2}(x,y)\right)\right)$$

$$\nabla_x^s G(x,y) = \frac{1}{4\pi}\frac{x \times y}{1 - \mathrm{x}\cdot y}$$

Point vortex dynamics on S^2:

$$u(p_j) = \frac{1}{4\pi}\sum_{\substack{i=1 \\ i\neq j}}^{n} \omega_i \frac{p_j \times p_i}{1 - \mathrm{p_i}\cdot p_j}$$

[Dri15] S. Dritschel, D. G.; Boatto. *The motion of point vortices on closed surfaces.*

The Sphere S^2

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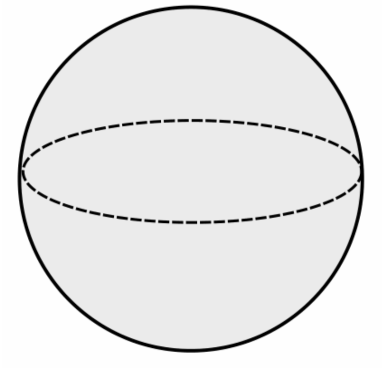

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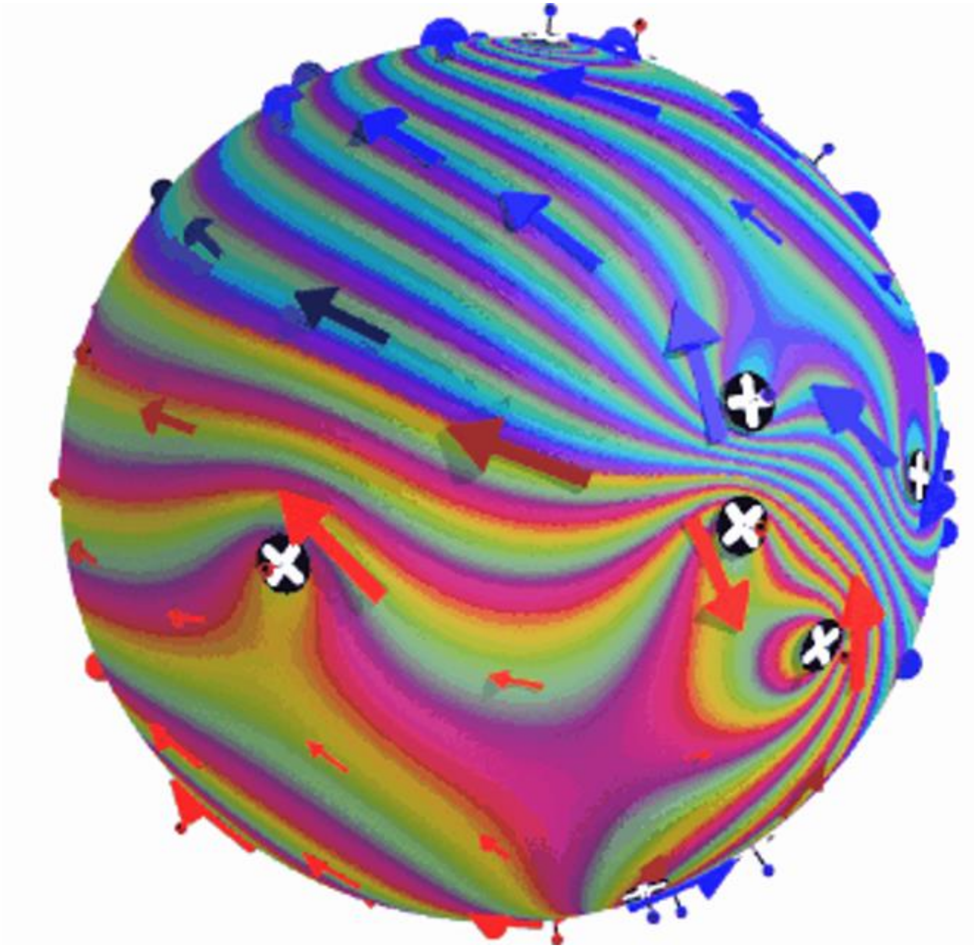
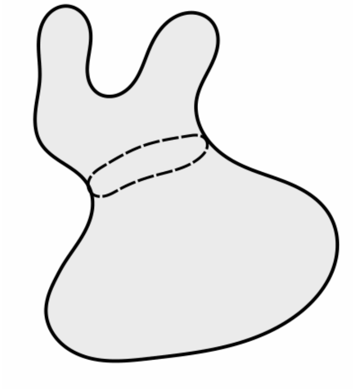


Closed Surface M

Closed Surface M



$G(x, y) = ???$

$\nabla_x^S G(x, y) = ???$

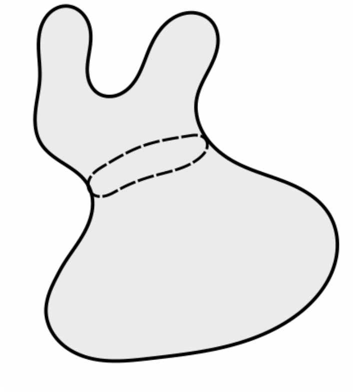
Point vortex dynamics on M:

$$u(p_j) = ???$$

Not clear!

Closed Surface M

Closed Surface M



$$\sum_{i=1}^{n} \omega_i = 0$$

[BK15] Stefanella Boatto and Jair Koiller.
Vortices on closed surfaces

Theorem (“*Conformal Metrics*”, [BK15]) . Within the context of $n \in \mathbb{N}$ point vortices p_i on S^2 with strengths ω_i, consider two metrics in the conformal class of S^2, related by a conformal factor h: $\tilde{g} = h^2 g$. The Hamiltonian $\widetilde{H}$ for the vortex system in the metric $\tilde{g}$ can be obtained from the Hamiltonian H in the metric g by adding two terms:

$$\widetilde{H} = H - \frac{1}{4\pi} \sum_{i=1}^{n} \omega_i^2 \ln(h(p_i)) - \left(\sum_{i=1}^{n} \omega_i \right) \frac{1}{\tilde{A}(S^2)} \sum_{i=1}^{n} \omega_i \Delta_g^{-1} h^2(p_i)$$

$\tilde{A}(S^2)$ describes the area in the $\tilde{g}$ metric.

Closed Surface M

[BK15] Stefanella Boatto and Jair Koiller.
Vortices on closed surfaces

Theorem (*"Conformal Metrics"*) if $\sum_{i=1}^{n} \omega_i = 0$

$$\widetilde{H} = H - \frac{1}{4\pi} \sum_{i=1}^{n} \omega_i^2 \ln\big(h(p_i)\big)$$

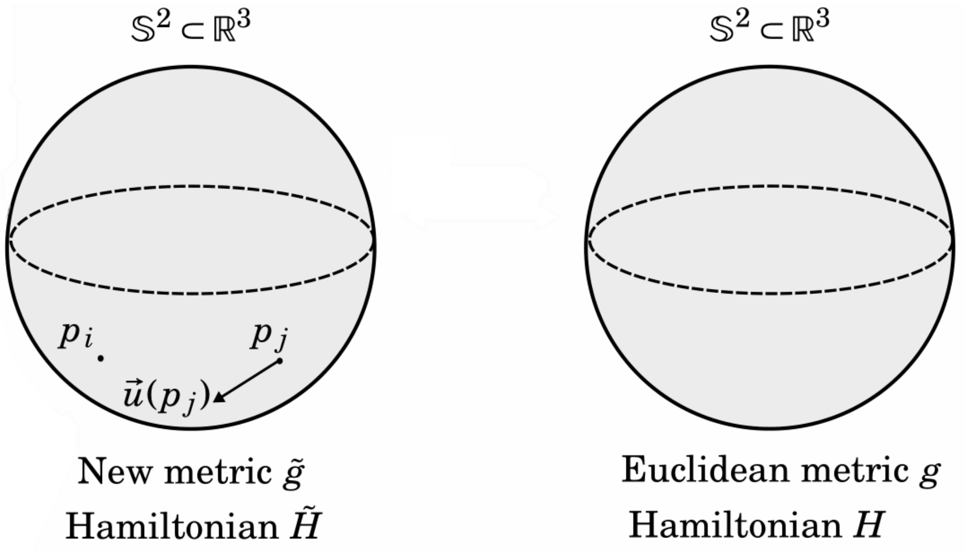
Given:

- two spheres
- different metrics

$$\tilde{g} = h^2 g$$

How does the Hamiltonian change

$$H \mapsto \widetilde{H}$$



Closed Surface M

[BK15] Stefanella Boatto and Jair Koiller.
Vortices on closed surfaces

Theorem (*"Conformal Metrics"*) if $\sum_{i=1}^{n} \omega_i = 0$

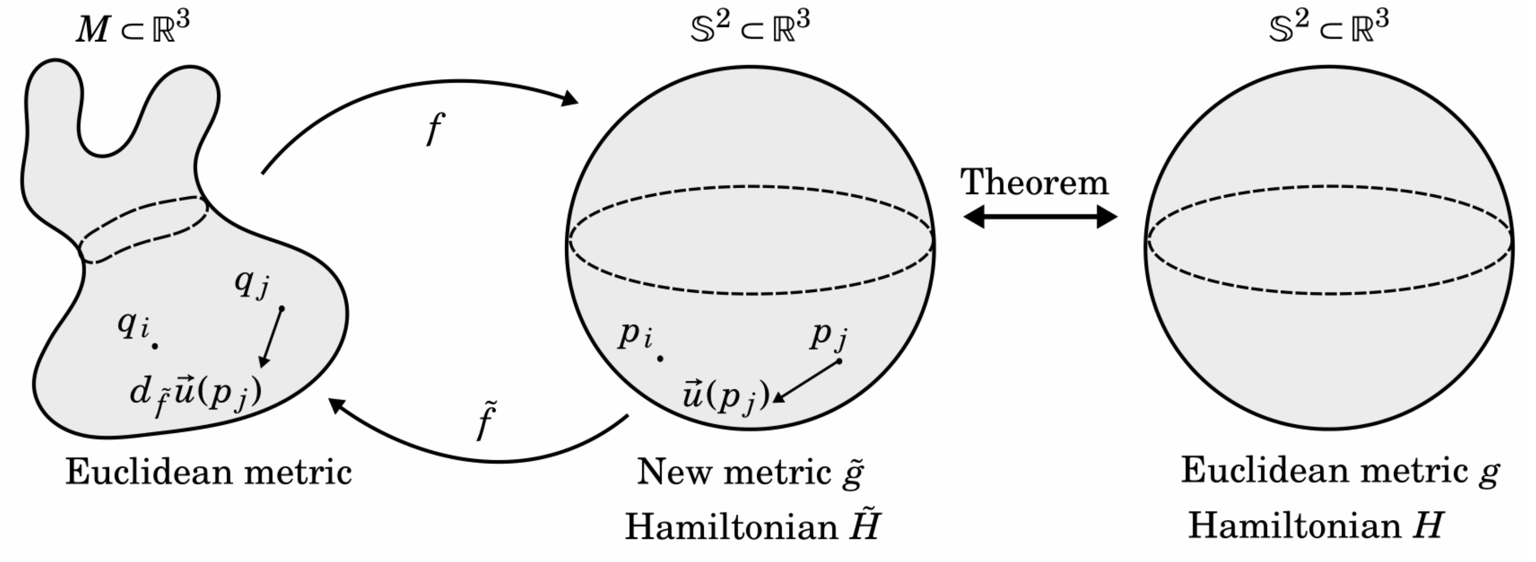
$$\widetilde{H} = H - \frac{1}{4\pi}\sum_{i=1}^{n} \omega_i^2 \ln\big(h(p_i)\big)$$

$$u(p_j) = -\frac{1}{\omega_j}\widetilde{\nabla}^S_{p_j}\widetilde{H}$$

$$\tilde{g} = h^2 g$$

$f: M \mapsto S^2$ conformal map

$h: M \mapsto \mathbb{R}$ conformal factor



Closed Surface M

$$u(p_j) = -\frac{1}{\omega_j}\widetilde{\nabla}^S_{p_j}\widetilde{H} = \frac{1}{h^2(p_j)}\left(-\frac{1}{\omega_j}\nabla^s_{p_j}H + \frac{1}{4\pi}\nabla^s_{p_j}\sum_{i=1}^{n}\omega_i^2 \ln(h(p_i))\right)$$

$$\widetilde{\nabla}^S_{p_j} = \frac{1}{h^2(p_j)}\nabla^s_{p_j}$$

$\tilde{g} = h^2 g$

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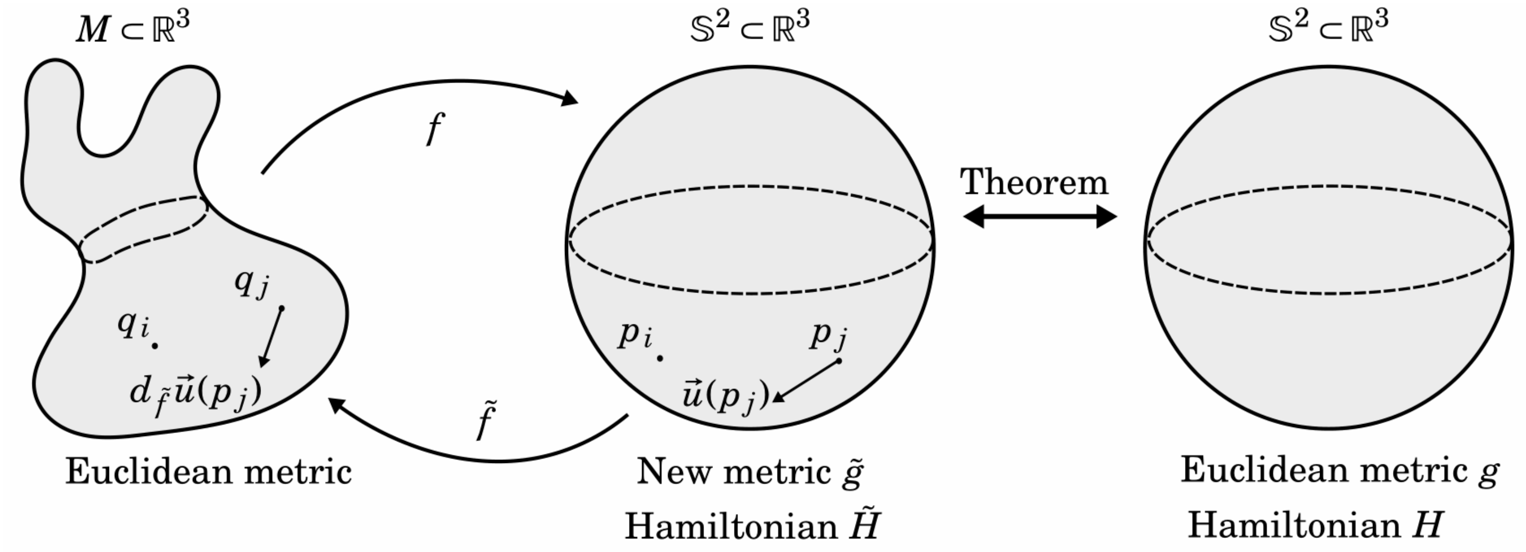

Closed Surface M

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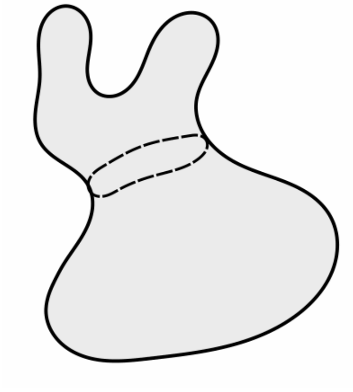

$$\sum_{i=1}^{n} \omega_i = 0$$

$f\colon M \mapsto S^2$ conformal map

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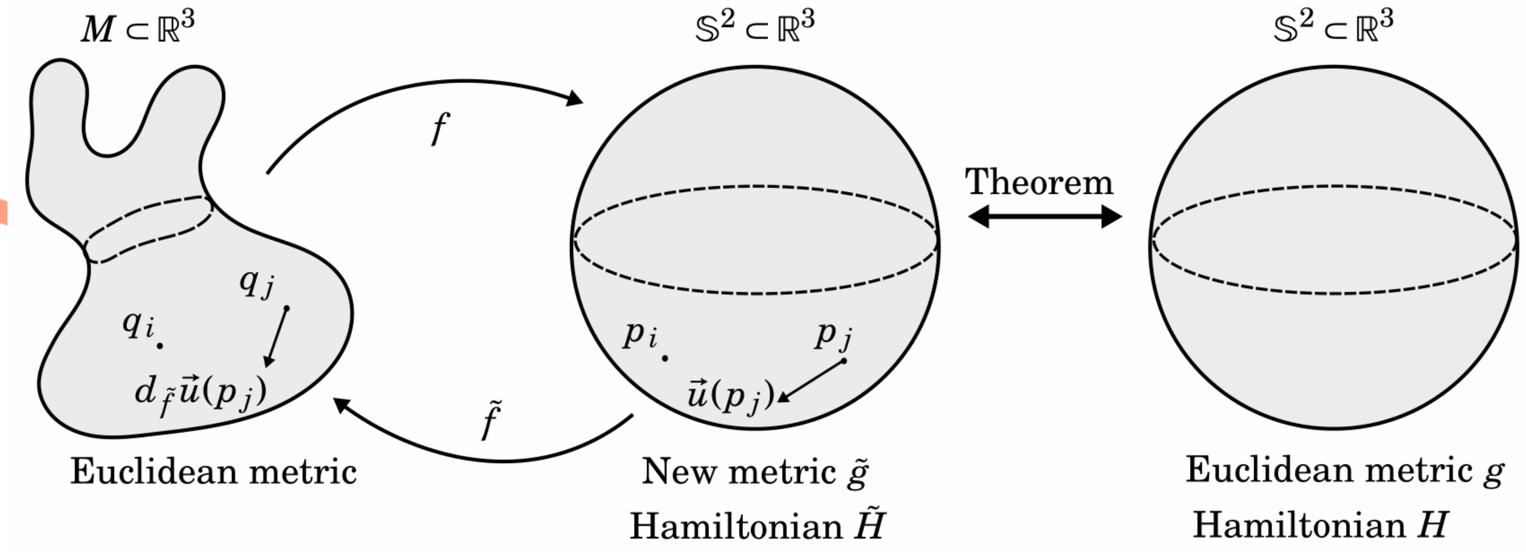
Point vortex dynamics on M:

$$u(p_j) = \frac{1}{4\pi h^2(p_j)} \left(\sum_{\substack{i=1 \\ i\neq j}}^{n} \omega_i \frac{p_j \times p_i}{1 - p_i \cdot p_j} + \frac{1}{h(p_j)} \omega_j^2 p_j \times \nabla_{p_j} h(p_j) \right)$$



Closed Surface M

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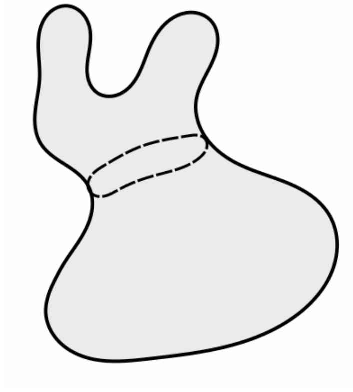

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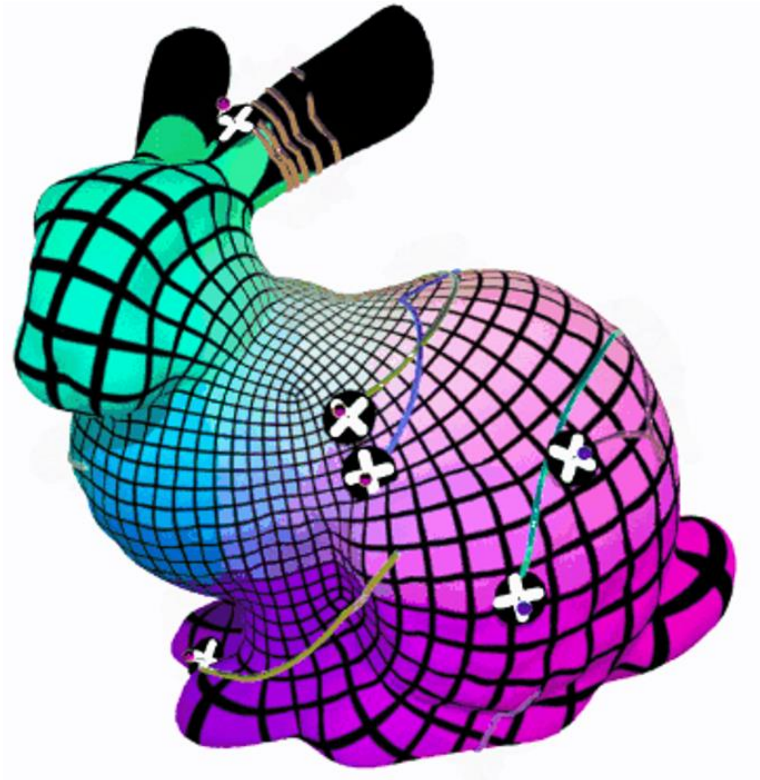
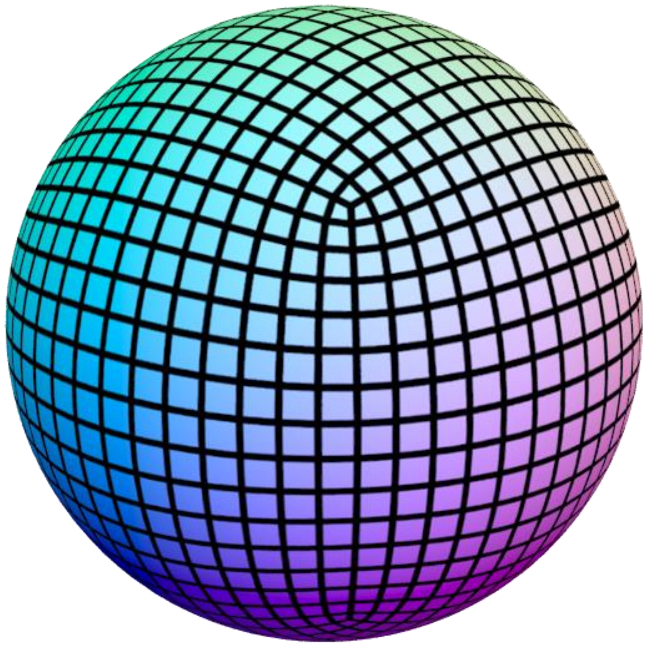
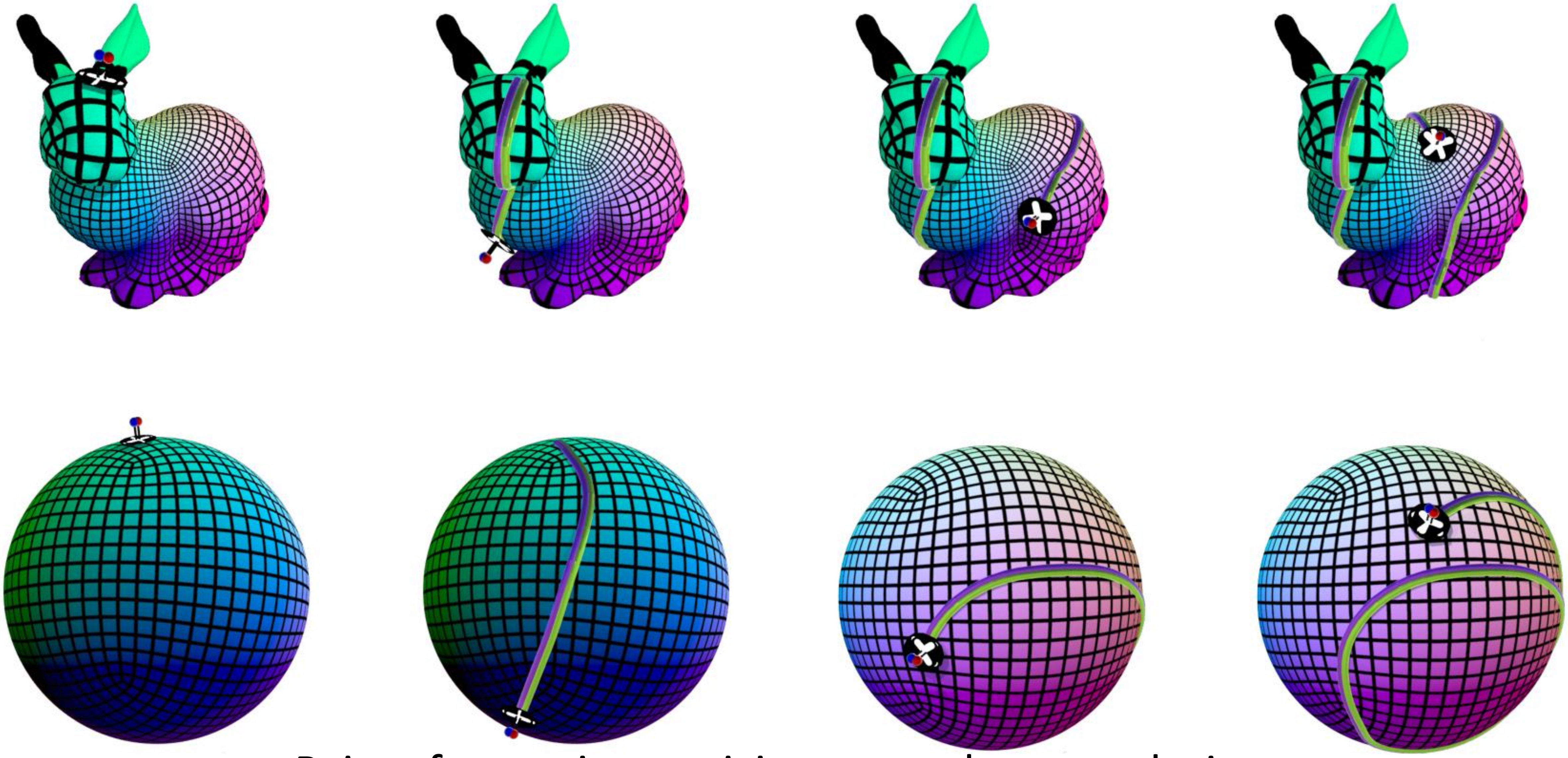


Geometry of the dynamics.



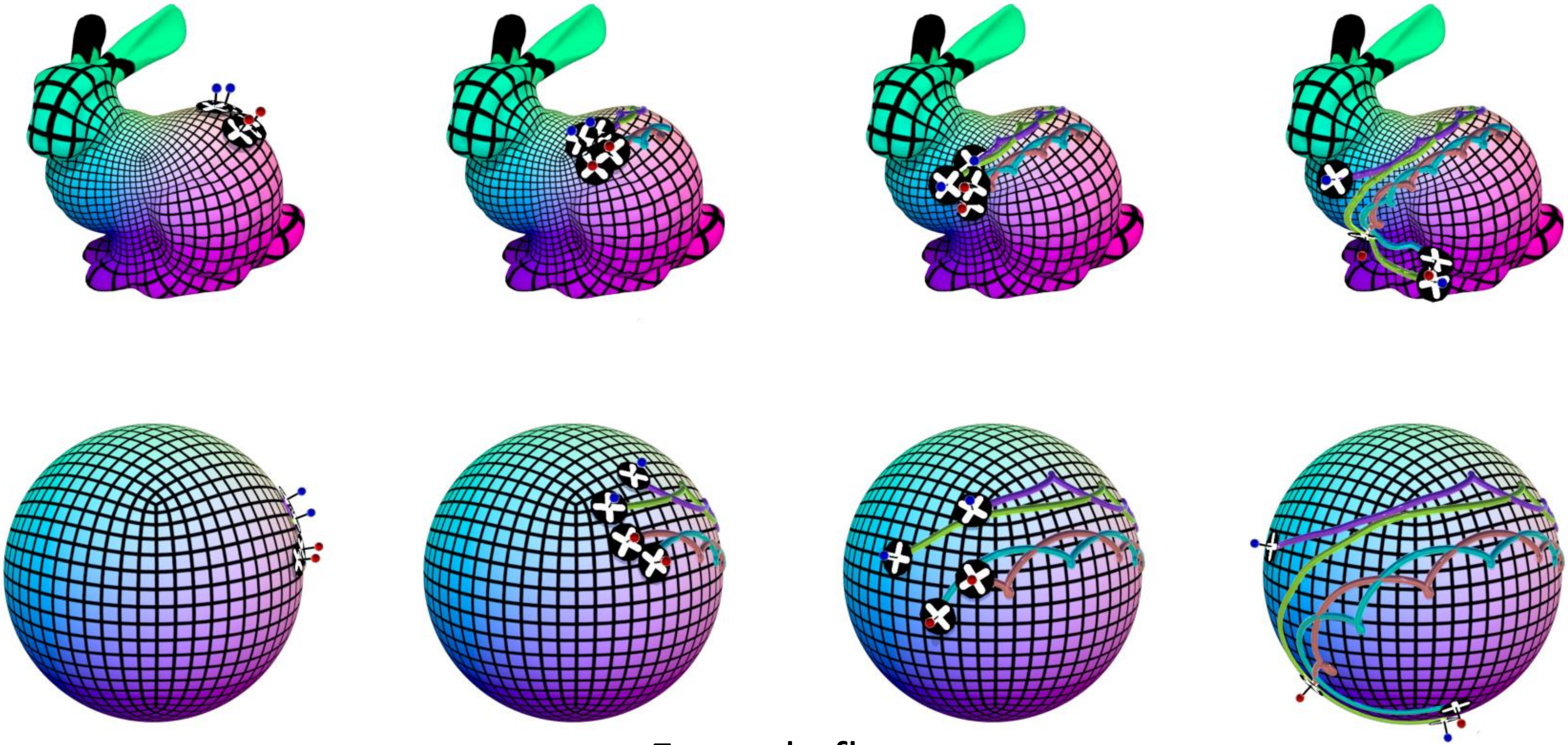
After conformal mapping.
The dynamics are computed here.

Closed Surface M



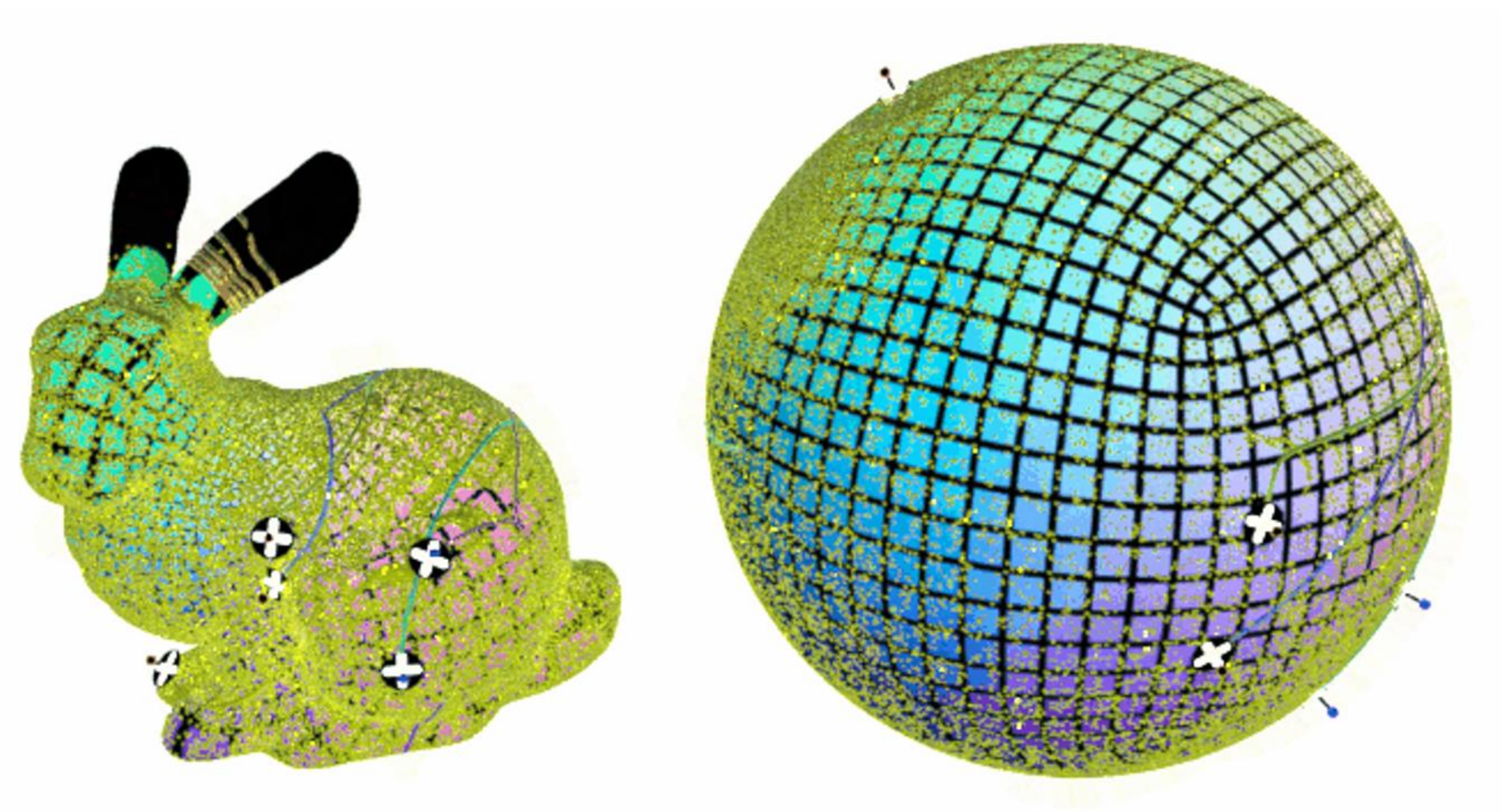
Pairs of opposite vorticity move along geodesics

Closed Surface M



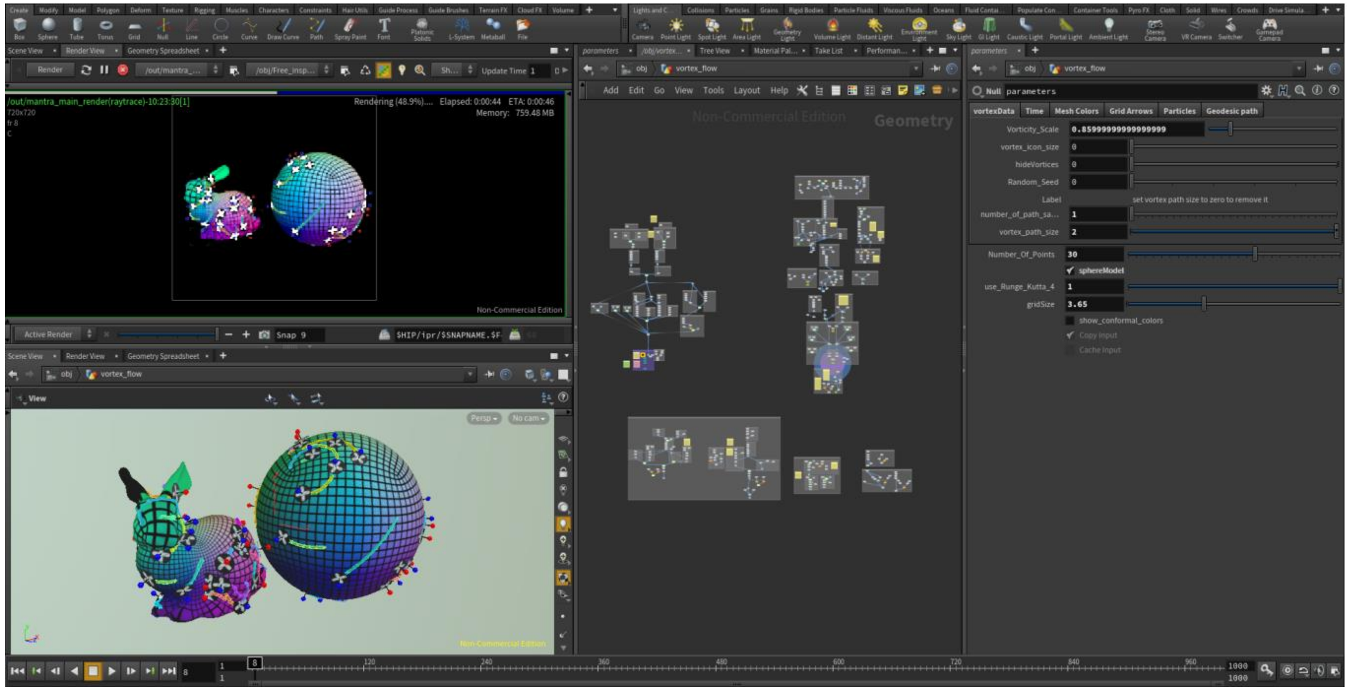
Example flow

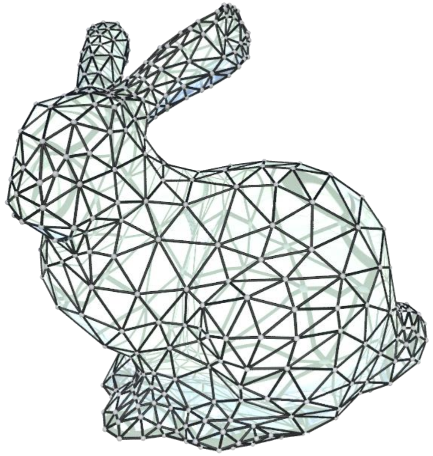
Closed Surface M

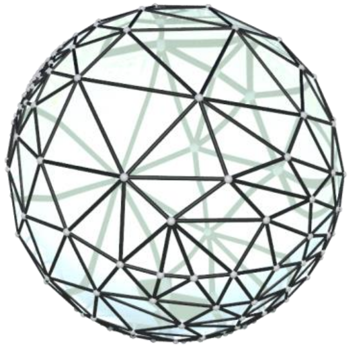


With passive particles

Implementation







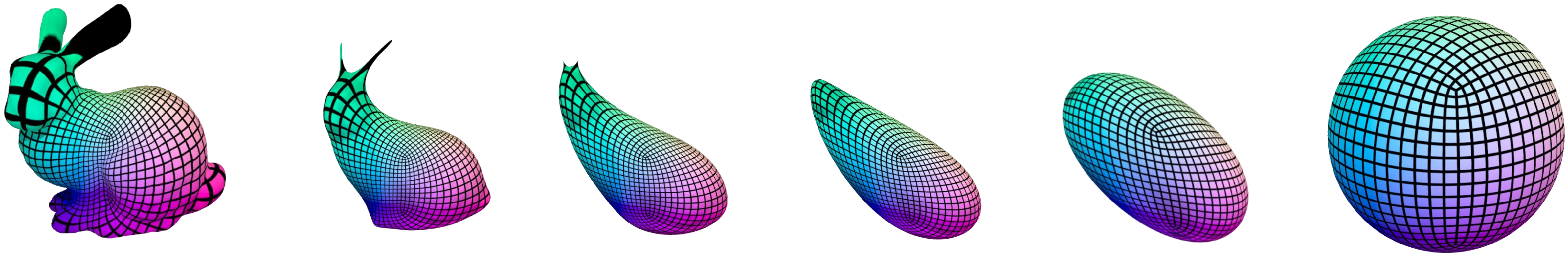
- Houdini VFX Software
- Python
- Runge-Kutta 4. Ordnung

Implementation

Conformal map from M to S^2

Modified mean curvature flow

$$f' = \Delta_0 f$$



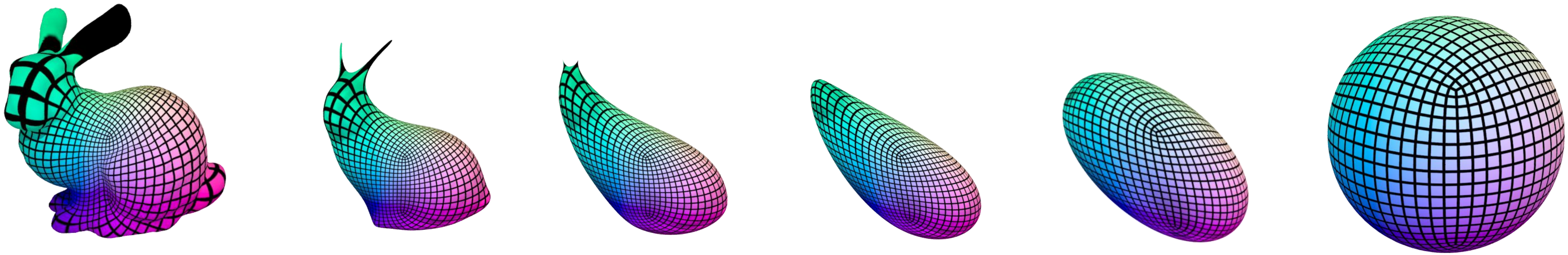
[KSBC12] Michael Kazhdan, Jake Solomon, and Mirela Ben-Chen. Can mean-curvature flow be modified to be non-singular?

Implementation

Conformal map from M to S^2

Modified mean curvature flow

$$(I - \epsilon\Delta_0)f_{n+1} = f_n$$



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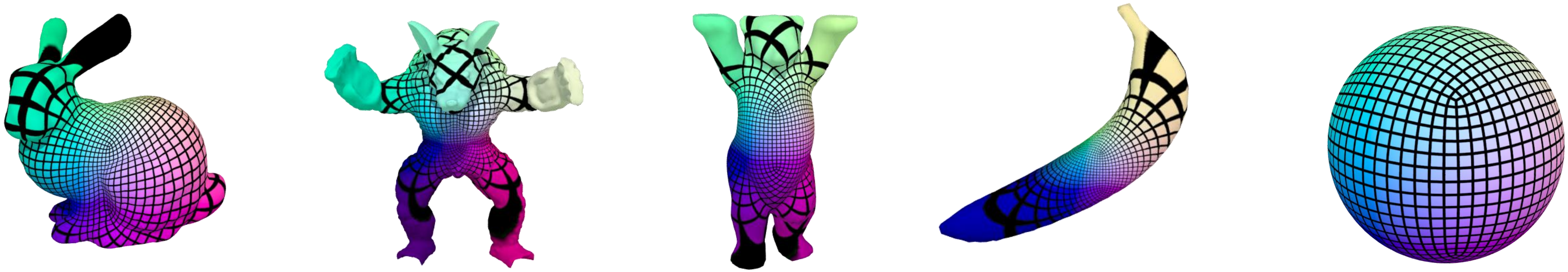

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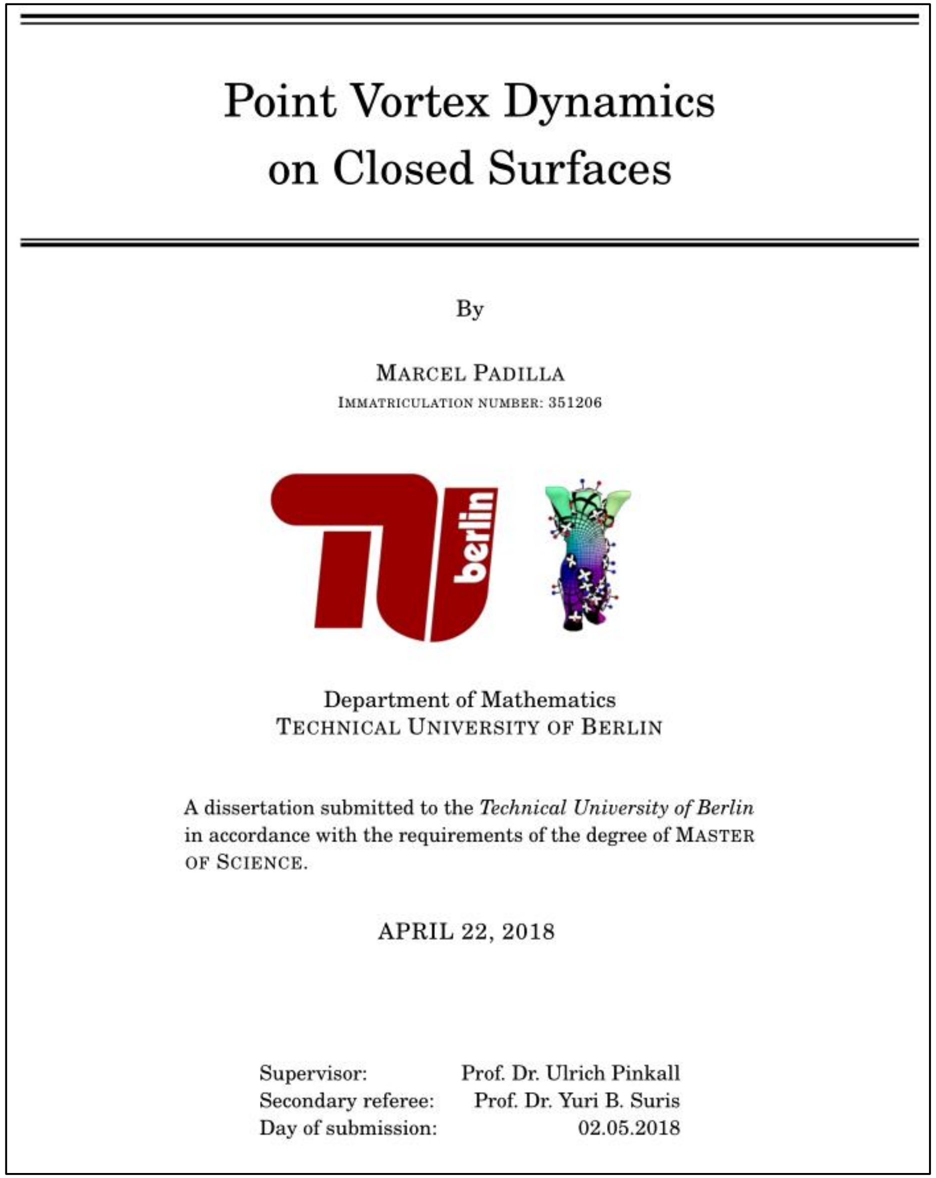
The End


Point Vortex Dynamics
on Closed Surfaces

By

MARCEL PADILLA
IMMATRICULATION NUMBER: 351206

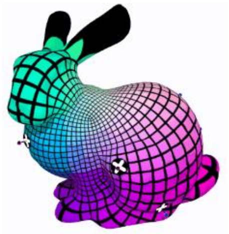
Department of Mathematics
TECHNICAL UNIVERSITY OF BERLIN

A dissertation submitted to the *Technical University of Berlin* in accordance with the requirements of the degree of MASTER OF SCIENCE.

APRIL 22, 2018

Supervisor:	Prof. Dr. Ulrich Pinkall
Secondary referee:	Prof. Dr. Yuri B. Suris
Day of submission:	02.05.2018

- Source-Code & PDF: http://page.math.tu-berlin.de/~padilla/
- Houdini Tutorials for Mathematicians: http://wordpress.discretization.de/houdini/



Thanks for your attention!